Electrostatic field of an uniformly charged sphere, force acting on the alpha particles and the electrostatic potential energy of the alpha particles in the Rutherford experiment

When Rutherford and co-workers (Geiger and Marsden) started their experiments the **Thomson model of the atom** was widely accepted. In this model the atoms consist of a heavy, positively charged sphere of radius R ("pudding") and the small, light, negatively charged electrons sit inside, making the whole electrically neutral (pudding model).

Rutherford studied the scattering of high energy alpha particles on a thin gold foil. When the positively charge alpha-particle approaches to a Thomson atom, the very light electrons will be displaced because of the electrostatic interaction – even they can be completely pulled away (the atom gets ionized). However, the electrons cannot influence the trajectory of the alpha particle, since their mass is really small, about 8000 times smaller then the alpha particle's mass. Therefore the trajectory of the alpha particle can be modified **only by the positively charged, heavy part** of the Thomson atom. Then, it is sufficient to analyse the interaction between a uniformly charged sphere with radius R, and a point like, positively charged particle. It is important to note that in this calculation only electrostatic interaction is considered. This means that the „pudding" is smooth; the alpha particle can even penetrate into it, if the electrostatic forces allow!

A charged sphere creates $\mathbf{E}(\mathbf{r})$ electrostatic field around itself. The force acting on a q charge in this field is: $\mathbf{F}(\mathbf{r}) = q\cdot\mathbf{E}(\mathbf{r})$.

In the following we determine the $\mathbf{E}(\mathbf{r})$ field, which is a vector field (bold face).

According to Gauss theorem $\oint \mathbf{E}(\mathbf{r})\cdot \mathrm{d}\mathbf{F} = \frac{1}{\varepsilon_0} Q$, where ε_0 is the permittivity of the vacuum; the integration should be done on a closed surface, and Q is the total electric charge inside this closed surface.

Because of the spherical symmetry the absolute value of the field depends only on the absolute value of the distance from the centre of the sphere (denoted by r), and the direction of the field vectors show outwards from the centre (if the sphere is positively charged). Therefore: $\mathbf{E}(\mathbf{r})\cdot \mathrm{d}\mathbf{F} = E(r)\cdot \mathrm{d}F$. This way the integral can easily be calculated for a sphere with radius r:
$\oint \mathbf{E}(\mathbf{r})\cdot \mathrm{d}\mathbf{F} = \oint E(r)\cdot \mathrm{d}F = E(r)\cdot 4\pi r^2$.

After having calculated the integral on the left hand side of the Gauss-theorem, let us consider the Q charge on the right hand side. This is the total charge inside the closed surface. Two cases should be distinguished:

a) If $r \geq R$, i.e. the integration sphere completely contains the „atom". In this case the charge will always be the same, independently of r: the total positive charge of the „pudding" Ze (here Z is the atomic number, e is the elementary charge).

   In this case we get $E(r)\cdot 4\pi r^2 = \frac{Ze}{\varepsilon_0}$, and this yields for the electrical field strength: $E(r) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Ze}{r^2}$. Taking into account that $\mathbf{F}(\mathbf{r}) = q\cdot\mathbf{E}(\mathbf{r})$, and that for alpha particles $q = 2e$, we get the well-known Coulomb force law: $\boxed{F(r) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Ze\cdot 2e}{r^2}}$.

b) The situation is different, if $r < R$, i.e. we integrate over a sphere which is „inside" the pudding-atom! In this case only a fraction of the total charge is inside the integrating sphere, therefore on the right hand side of the Gauss-theorem a smaller fraction of the total charge should appear. If the „pudding" is uniformly charged, the charge inside the sphere will be proportional to the volume of the sphere. Therefore we should write to the right hand side of the Gauss-theorem the following charge: $Q = Ze \cdot \dfrac{\frac{4\pi}{3} r^3}{\frac{4\pi}{3} R^3}$. The Gauss-theorem becomes: $E(r) \cdot 4\pi r^2 = \dfrac{Ze}{\varepsilon_0} \cdot \dfrac{r^3}{R^3}$. Finally, we get for the electrostatic field strength: $E(r) = \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{Ze}{R^3} \cdot r$.

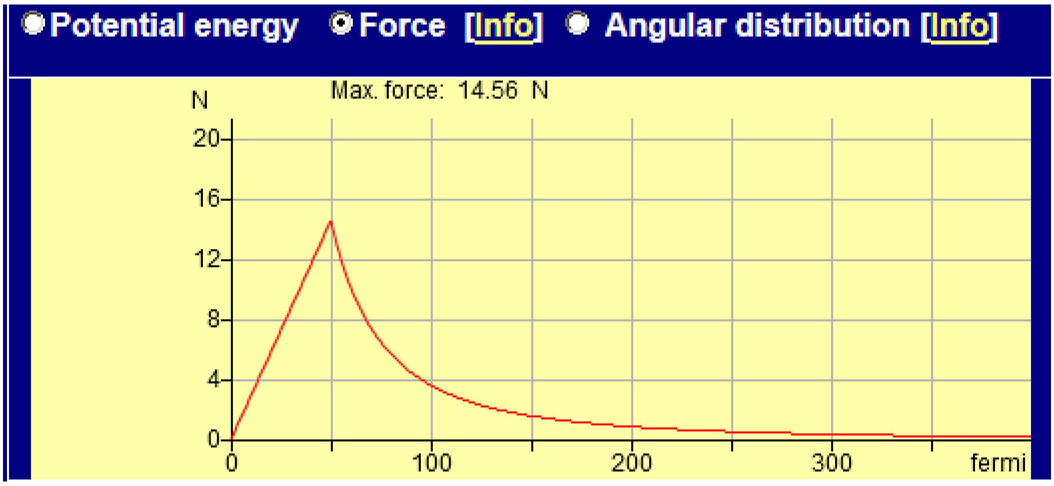
The force acting on the alpha particle becomes: $\boxed{F(r) = \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{Ze \cdot 2e}{R^3} \cdot r}$.

Notice, that the force vanishes at $r = 0$, in the centre. If you think about it, it makes sense: because of the complete spherical symmetry the resulting electrostatic force should be zero in the centre.

Summarized: the force acting on an alpha particle at an *r* distance from the centre of the sphere:

$$\boxed{F(r) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{Ze \cdot 2e}{R^3} \cdot r, & \text{ha } r < R \\ \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{Ze \cdot 2e}{r^2}, & \text{ha } r \geq R \end{cases}}.$$

This is shown in the following picture (for a gold atom sphere with R = 50 fm), and this can be seen also in the „**Force**" window of the simulation.



Knowing the force law the interaction potential can easily be deduced.

In the conservative electrostatic field the force is the negative gradient of the potential: $\mathbf{E}(\mathbf{r}) = -\mathrm{grad}U(\mathbf{r})$. Since we have spherical symmetry, everything depends only on the absolute value of the radius, therefore $E(r) = -\dfrac{\partial}{\partial r}U(r)$. The potential can be determined from this equation by simple integration, if $E(r)$ is known.

We fix the 0 value of the potential in the infinity, therefore $U(r) = -\int_{\infty}^{r} E(r')\mathrm{d}r'$. Since the force law is different outside and inside the „pudding", this integral is also composed of two parts. If $r \geq R$, then $U(r) = -\frac{Ze}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{1}{r'^2}\mathrm{d}r' = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Ze}{r}$.

If $r < R$, then we should continue the integration „inward" from r=R:

$$U(r) = U(R) - \frac{1}{4\pi\varepsilon_0}\cdot\frac{Ze}{R^3}\int_R^r r'\mathrm{d}r' = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Ze}{R^3}\left(R^2 - \left[\frac{r^2}{2}\right]_R^r\right) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Ze}{R^3}\left(R^2 - \frac{r^2}{2} + \frac{R^2}{2}\right)$$

Finally we get: $\boxed{U(r) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\cdot\frac{Ze}{r}, & \text{ha } r \geq R \\ \frac{1}{4\pi\varepsilon_0}\cdot\frac{Ze}{2R^3}\left(3 - \frac{r^2}{R^2}\right), & \text{ha } r < R \end{cases}}$.

The potential energy of the alpha particle: $E_{pot}(r) = 2e \cdot U(r)$

This is shown in the following picture (for a gold atom sphere with R = 50 fm), and this can be seen also in the „**Potential energy**" window of the simulation. The horizontal purple line shows the total energy of the alpha particle, which is $E = E_{kin} + E_{pot}$ (E = 3 MeV in the picture). Since the E_{kin} kinetic energy cannot be negative, the alpha-particle can approach the nucleus only while $E_{pot}<E$, i.e. the horizontal line is above the potential energy (about 75 fm).