Nuclear Physics

Practice 13

Exercise 1: Neutron slowing down theory

Let us consider the E'/E ratio from the previous calculations with a neutron scattering on a nucleus with A mass number:

$$\frac{E'}{E} = \frac{m_A^2 + m_n^2 + 2m_n m_A \cos\theta_{CM}}{(m_A + m_n)^2}$$

$$\frac{E'}{E} = \frac{A^2 + 1 + 2A\cos\theta_{CM}}{(A+1)^2}$$

We will define the α parameter as

$$\alpha = \frac{(A-1)^2}{(A+1)^2}$$

The energy ratio can be expressed using this parameter:

$$\frac{E'}{E} = \frac{1}{2}\left[(1+\alpha) + (1-\alpha)\cos\theta_{CM}\right]$$

From the above equation it can be easily seen that the energy of the scattered neutron can be $\alpha E \le E' \le E$. The maximum change in the logarithm of the energy is independent of the energy itself:

$$0 \le \ln\left(\frac{E}{E'}\right) \le \ln\left(\frac{1}{\alpha}\right)$$

Let us denote the cosine of the scattering angle in the CM frame with $\mu=\cos(\theta_{CM})$. In the case of isotropic scattering the solid angle in which the neutron is scattered, Ω takes its values with uniform distribution in 4π direction. The probability of scattering to an angle between μ and $\mu+d\mu$ can be calculated as the following:

$$f(\mu)d\mu = \frac{d\Omega}{4\pi} = \frac{2\pi \cdot d\mu}{4\pi} = \frac{1}{2}d\mu$$

where $f(\mu)$ is the probability density function of μ. Using the relation between the final energy E' and $\cos(\theta_{CM})$ the probability of scattering to an energy between E' and $E'+dE'$ can be expressed:

$$g(E,E')dE' = f(\mu)d\mu = f(\mu)\frac{d\mu}{dE'}dE' = \frac{1}{2}\cdot\frac{2}{E(1-\alpha)}dE' = \frac{1}{E(1-\alpha)}dE'$$

where $g(E,E')$ is the probability density function of E'. Since this does not depend on the value of E' therefore in the case of isotropic scattering the energy of the scattered neutron E' also takes its values with uniform distribution between αE and E.

In slowing down theory we usually use the neutron lethargy instead of the neutron energy:

$$u = \ln\left(\frac{E_0}{E}\right)$$

where E_0 is some arbitrary upper limit. We can also introduce the average logarithmic energy decrease:

$$\xi = \int_{\alpha E}^{E} \ln\left(\frac{E}{E'}\right) g(E, E') dE' = 1 + \frac{\alpha}{1-\alpha}\ln(\alpha) \approx \frac{2}{A + 2/3}$$

If ξ is greater, the neutrons can slow down faster, since lower number of collisions is needed to reach the thermal energy range (0.025 eV) from the average energy of neutrons which are created in fission (2 MeV):

$$\frac{\ln\left(\frac{2\ \text{MeV}}{0.025\ \text{eV}}\right)}{\xi} = \frac{18.2}{\xi}$$

Those materials which can effectively slow neutrons down are called moderators. These are isotopes with low mass number some examples include: H, H_2O, D_2O, Be, C. The number of collisions needed to slow down neutrons to the thermal energy range is 18 in water, while it would be more than 2000 in ^{238}U.

Exercise 2: Proton-proton collision in accelerators

Calculate the available energy after the fusion of two colliding protons if a) one of the protons has 14 TeV energy and the other one is at rest and b) both of the protons have 7 TeV energy.

Solution:

a) The total energy before the collision (relativistic formula):

$$E = \sqrt{(p_1 c)^2 + (mc^2)^2} + mc^2$$

The momentum of the product (which has approximately 2m mass) is equal to the momentum of the first proton due to momentum conservation law, therefore the total energy after the collision will be:

$$E' = \sqrt{(p_1 c)^2 + (2mc^2)^2} + \Delta E_1$$

The available energy can be calculated from the energy conservation:

$$\sqrt{(p_1 c)^2 + (mc^2)^2} + mc^2 = \sqrt{(p_1 c)^2 + (2mc^2)^2} + \Delta E_1$$

$$\Delta E_1 = \sqrt{(p_1 c)^2 + (mc^2)^2} + mc^2 - \sqrt{(p_1 c)^2 + (2mc^2)^2} = p_1 c \sqrt{1 + \left(\frac{mc^2}{p_1 c}\right)^2} + mc^2 - p_1 c \sqrt{1 + 4\left(\frac{mc^2}{p_1 c}\right)^2}$$

Since $(mc^2/p_1 c)^2 << 1$, we can approximate the square roots with their first order Taylor-expansion and get the following released energy:

$$\Delta E_1 \approx p_1 c \cdot \left[1 + \frac{1}{2}\left(\frac{mc^2}{p_1 c}\right)^2\right] + mc^2 - p_1 c \cdot \left[1 + 2\left(\frac{mc^2}{p_1 c}\right)^2\right] \cong mc^2$$

b) The total energy before the collision:

$$E = \sqrt{(p_2 c)^2 + (mc^2)^2} + \sqrt{(p_2 c)^2 + (mc^2)^2} = 2\sqrt{(p_2 c)^2 + (mc^2)^2}$$

Since the total momentum in this case was zero before the collision, it will also be zero after the collision. Therefore we can write for the total energy after the collision:

$$E' = 2mc^2 + \Delta E_2$$

The available energy can be calculated again from the energy conservation (using the same Taylor-expansion):

$$\Delta E_2 = 2\sqrt{(p_2 c)^2 + (mc^2)^2} - 2mc^2 \approx 2 p_2 c$$

The ratio between the available energies in the first and second case is therefore:

$$\frac{\Delta E_2}{\Delta E_1} = \frac{2 p_2 c}{mc^2} \approx 14900$$