Nuclear Physics

Practice 12

Exercise 1: Energy loss in elastic scattering

Let us examine the equation concerning the projectile velocity after the collision (see Appendix):

$$v'^2_p = (v_p - v_{CM})^2 + v^2_{CM} + 2(v_p - v_{CM})v_{CM}\cos\theta_{CM}$$

$$v'^2_p = v^2_p - 2v_p v_{CM} + 2v^2_{CM} + 2(v_p - v_{CM})v_{CM}\cos\theta_{CM}$$

By rearranging this equation we can obtain the ratio of the final and initial velocity squares, which is equal to the ratio of the projectile energies:

$$\frac{E'}{E} = \frac{v'^2_p}{v^2_p} = 1 - \frac{2m_p}{m_t + m_p} + \frac{2m^2_p}{(m_t + m_p)^2} + \left(\frac{2m_p}{m_t + m_p} - \frac{2m^2_p}{(m_t + m_p)^2}\right)\cos\theta_{CM} = \%$$

$$\% = \frac{m^2_t + 2m_p m_t + m^2_p - 2m_p m_t - 2m^2_p + 2m^2_p + (2m_p m_t + 2m^2_p - 2m^2_p)\cos\theta_{CM}}{(m_t + m_p)^2} = \%$$

$$\% = \frac{m^2_t + m^2_p + 2m_p m_t \cos\theta_{CM}}{(m_t + m_p)^2}$$

It can be easily seen that the maximal energy loss corresponds to 180°, where $\cos\theta_{CM} = -1$. The maximal energy loss will be therefore:

$$\Delta E_{\max} = E - E' = \left[1 - \frac{m^2_t + m^2_p - 2m_p m_t}{(m_t + m_p)^2}\right]\cdot E = \left[1 - \frac{(m_t - m_p)^2}{(m_t + m_p)^2}\right]\cdot E = \frac{4m_p m_t}{(m_t + m_p)^2}\cdot E$$

if $m_p \ll m_t$:

$$\Delta E_{\max} = \frac{4m_p m_t}{(m_t + m_p)^2}\cdot E \approx \frac{4m_p m_t}{m^2_t}\cdot E = \frac{4m_p}{m_t}\cdot E$$

if $m_p \approx m_t$:

$$\Delta E_{\max} = \frac{4m_p m_t}{(m_t + m_p)^2}\cdot E \approx \frac{4m_p m_p}{(m_p + m_p)^2}\cdot E = E$$

if $m_p \gg m_t$:

$$\Delta E_{\max} = \frac{4m_p m_t}{(m_t + m_p)^2}\cdot E \approx \frac{4m_p m_t}{m^2_p}\cdot E = \frac{4m_t}{m_p}\cdot E$$

Exercise 2: Measurement of the neutron mass from scattering experiment results

Chadwick estimated the mass of the neutron from measurements of the elastic scattering of neutrons by hydrogen and nitrogen. The maximum velocity of the recoiling protons was observed $3.4 \cdot 10^7$ m/s while the maximum recoil velocity of the nitrogen ions was $4.6 \cdot 10^6$ m/s. Estimate the mass of the neutron. What was the energy of the incident neutrons?

Solution:

According to the results of the previous example we can write two equations:

$$\frac{1}{2} m_p v_p^2 = \frac{4 m_n m_p}{(m_p + m_n)^2} \cdot E$$

$$\frac{1}{2} m_N v_N^2 = \frac{4 m_n m_N}{(m_N + m_n)^2} \cdot E$$

Let us divide the first equation with the second one to eliminate the unknown incident energy of the neutron:

$$\frac{v_p^2}{v_N^2} = \frac{(m_N + m_n)^2}{(m_p + m_n)^2}$$

$$\frac{v_p}{v_n} = \frac{m_N + m_n}{m_p + m_n}$$

By rearranging this the mass of the neutron can be expressed (m_N=14.003074 u, m_p=1.007276 u):

$$m_n = \frac{m_p \dfrac{v_p}{v_n} - m_N}{1 - \dfrac{v_p}{v_n}} \cong 1.026 \text{ u}$$

which is somewhat higher than the exact neutron mass. The incident neutron energy can be calculated by substituting back to either one of the two equations:

$$E = \frac{(m_p + m_n)^2 v_p^2}{8 m_n} \cong 5.64 \text{ MeV}$$

Exercise 3: Neutron slowing down theory

Let us consider the *E*'/*E* ratio from the previous calculations with a neutron scattering on a nucleus with *A* mass number:

$$\frac{E'}{E} = \frac{m_A^2 + m_n^2 + 2m_n m_A \cos\theta_{CM}}{(m_A + m_n)^2}$$

$$\frac{\mathrm{E'}}{\mathrm{E}} = \frac{\mathrm{A}^2 + 1 + 2\mathrm{A}\cos\theta_{\mathrm{CM}}}{(\mathrm{A}+1)^2}$$

We will define the α parameter as

$$\alpha = \frac{(A-1)^2}{(A+1)^2}$$

The energy ratio can be expressed using this parameter:

$$\frac{E'}{E} = \frac{1}{2}\left[(1+\alpha) + (1-\alpha)\cos\theta_{CM}\right]$$

From the above equation it can be easily seen that the energy of the scattered neutron can be $\alpha E \le E' \le E$. The maximum change in the logarithm of the energy is independent of the energy itself:

$$0 \le \ln\left(\frac{E}{E'}\right) \le \ln\left(\frac{1}{\alpha}\right)$$

In the case of isotropic scattering the energy of the scattered neutron, *E*' takes its values with uniform distribution between αE and *E*. The probability of scattering to an energy between E' and E'+dE' can be written as the following (i.e. the probability density function of *E*'):

$$g(E,E')dE' = \frac{1}{E(1-\alpha)}dE'$$

which does not depend on the value of *E*'. In slowing down theory we usually use the neutron lethargy instead of the neutron energy:

$$u = \ln\left(\frac{E_0}{E}\right)$$

where E_0 is some arbitrary upper limit. We can also introduce the average logarithmic energy decrease:

$$\xi = \int_{\alpha E}^{E} \ln\left(\frac{E}{E'}\right) g(E,E')dE' = 1 + \frac{\alpha}{1-\alpha}\ln(\alpha) \approx \frac{2}{A + 2/3}$$

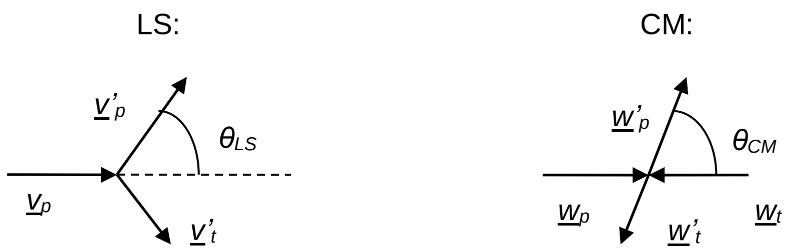
If ξ is greater, the neutrons can slow down faster, since lower number of collisions are needed to reach the thermal energy range (0.025 eV) from the average energy of neutrons which were created in fission (2 MeV):

$$\frac{\ln\left(\frac{2\ \mathrm{MeV}}{0.025\ \mathrm{eV}}\right)}{\xi} = \frac{18.2}{\xi}$$

Those materials which can effectively slow neutrons down are called moderators. These are isotopes with low mass number some examples include: H, H_2O, D_2O, Be, C. The number of collisions needed to slow down neutrons to the thermal energy range is 18 in water, while it would be more than 2000 in ^{238}U.

Appendix: Scattering angles in laboratory system and center of mass system

For a comparison between theory and experiment in nuclear reactions, it is necessary to have cross-sections in the same reference frame. The measured cross-section is obtained in the laboratory frame, where we assume that the target is at rest. From the theoretical point of view, however, it is usually advantageous to introduce the center of mass frame. Let us consider now the simplest example, which is the elastic scattering:



where v_p is the initial velocity of the projectile particle in the laboratory frame (LS), v'_p and v'_t are the velocities of the projectile and target after the collision in the LS frame, and w_p, w_t, w'_p, w'_t are the corresponding velocities in the center of mass (CM) frame. We assume that v_t=0.

Introducing the velocity of the center of mass, the following relations apply between the velocities in the LS and CM frame:

$$v_{CM} = \frac{m_p v_p}{m_t + m_p}$$

$$\underline{w}_p = \underline{v}_p - \underline{v}_{CM}$$

$$\underline{w}_t = \underline{v}_t - \underline{v}_{CM} = -\underline{v}_{CM}$$

$$\underline{w}'_p = \underline{v}'_p - \underline{v}_{CM}$$

$$\underline{w}'_t = \underline{v}'_t - \underline{v}_{CM}$$

In the CM frame, the overall momentutm of the projectile and target is zero either before and after the collision (since the velocities are in the opposite direction, we can calculate with their absolute values):

$$m_p w_p - m_t w_t = 0$$

$$m_p w'_p - m_t w'_t = 0$$

from these equations we get the velocity of the target before and after the collision:

$$w_t = \frac{m_p}{m_t} w_p$$

$$w'_t = \frac{m_p}{m_t} w'_p$$

The kinetic energy is conserved, which gives us:

$$\frac{1}{2} m_p w_p^2 + \frac{1}{2} m_t w_t^2 = \frac{1}{2} m_p w'^2_p + \frac{1}{2} m_t w'^2_t$$

$$\frac{1}{2} m_p w_p^2 + \frac{1}{2} m_t \frac{m_p^2}{m_t^2} w_p^2 = \frac{1}{2} m_p w'^2_p + \frac{1}{2} m_t \frac{m_p^2}{m_t^2} w'^2_p$$

$$w_p = w'_p$$

$$w_t = w'_t$$

This means that in the CM frame only the direction of the velocities changes, their absolute value remains the same. From the relation between the final projectile velocity in the LS and CM frame, one can express the scattering angle in the center of mass system:

$$\underline{v}'_p = \underline{w}'_p + \underline{v}_{CM}$$

$$v'^2_p = w'^2_p + v^2_{CM} + 2w'_p\, v_{CM} \cos\theta_{CM}$$

$$v'^2_p = (v_p - v_{CM})^2 + v^2_{CM} + 2(v_p - v_{CM}) v_{CM} \cos\theta_{CM}$$

$$\cos\theta_{CM} = \frac{v'^2_p - v^2_p - 2v^2_{CM} + 2v_p v_{CM}}{2(v_p - v_{CM}) v_{CM}} = \frac{v'^2_p - v^2_p}{2(v_p - v_{CM}) v_{CM}} - 1 = \frac{v'^2_p - v^2_p}{2\left(1 - \frac{m_p}{m_t + m_p}\right) \frac{m_p}{m_t + m_p} v^2_p} - 1$$

$$\cos\theta_{CM} = \frac{v'^2_p - v^2_p}{2 \frac{m_t m_p}{(m_t + m_p)^2} v^2_p}$$

The relation between the scattering angles in the LS and CM frames can be determined with the examination of the z direction:

$$v'_p \cos\theta_{LS} = w'_p \cos\theta_{CM} + v_{CM} = (v_p - v_{CM}) \cos\theta_{CM} + v_{CM}$$

$$\cos\theta_{LS} = \frac{v_p}{v'_p}\left[\left(1 - \frac{m_p}{m_t + m_p}\right)\cos\theta_{CM} + \frac{m_p}{m_t + m_p}\right] = \frac{v_p}{v'_p}\left[\frac{m_t \cos\theta_{CM} + m_p}{m_t + m_p}\right]$$

This result shows us that if $m_p \ll m_t$ then the scattering angles in the LS and CM frame are very close and that if $m_p \gg m_t$ then the scattering angle in the LS system will be close to zero.