Nuclear Physics

Practice 10

Remark: A brief note about the Woods-Saxon potential

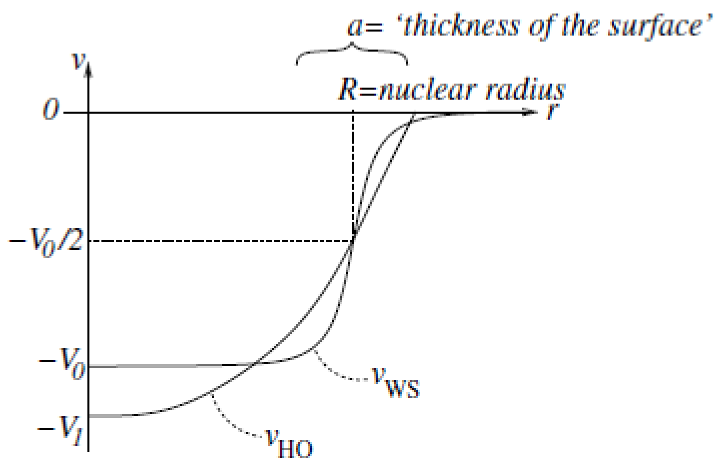
The more realistic potential used in the shell-model is the Woods-Saxon potential (which is an approximation to the potential obtained with the Hartree-Fock method):

$$V_{WS}(r) = \frac{-V_0}{1+\mathrm{e}^{(r-R)/a}}$$

where R is the nuclear radius and a is the surface diffuseness. Typical values for the parameters are:

$$R = r_0 A^{1/3}, \quad a = 0.67 \text{ fm}, \quad V_0 = 51 \pm 33 \frac{N-Z}{A} \text{ MeV}$$

The plus sign in V_0 is for neutrons and the minus is for protons. The relation between the Woods-Saxon potential and the equivalent harmonic oscillator potential can be seen in the following figure:

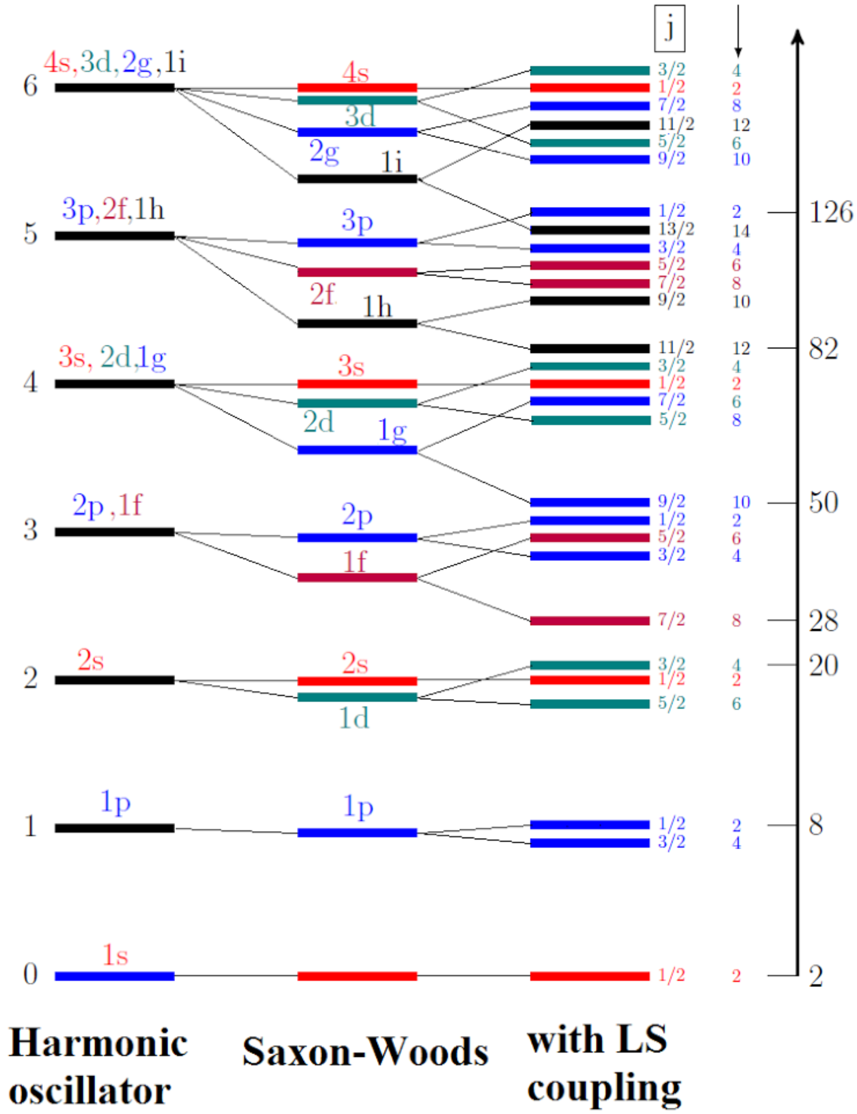


The corrections *compared to the harmonic oscillator potential* are

a) repulsive effect for short and large distances → push up small l orbits
b) attractive effect for intermediate → push down large l orbits

However, the Schrödinger-equation cannot be solved analitically for the WS-potential, only numerical solutions are provided. The equivalent harmonic oscillator potential reproduces the wave functions of the Woods–Saxon potential quite well near the bottom of the wells, but when approaching zero energy, the differences grow. Near zero energy the Woods–Saxon wave functions have a long tail extending far beyond the nuclear radius R, whereas the harmonic oscillator wave functions decrease sharply beyond the potential wall.

The correct level-scheme was obtained by calculating the spin-orbit interaction with the WS-potential (remember that the spin-orbit potential was proportional to $dV(r)/dr$):



Exercise 1: Quadrupole moments

Compute the expected shell model quadrupole moment of ^{209}Bi ($9/2^-$) and compare it with the experimental value, -0.37 barn.

Solution:

The quadrupole moment of the unpaired nucleon according to the shell model is:

$$Q_p = <\Psi_j \,|\, 3z^2 - r^2 \,|\, \Psi_j> = -\frac{2j-1}{2(j+1)} <\Psi_j \,|\, r^2 \,|\, \Psi_j> = -\frac{2j-1}{2(j+1)} \cdot \frac{3}{5} R^2 = -\frac{2j-1}{2(j+1)} \cdot \frac{3}{5} r_0^2 A^{2/3}$$

where $r_0 = 1.2 \text{ fm}$.

Substuting back $I^\pi = 9/2^-$ into the formula:

$$Q_p = -\frac{2 \cdot \frac{9}{2} - 1}{2\left(\frac{9}{2} + 1\right)} \cdot \frac{3}{5} (1.2 \cdot 10^{-15})^2 \cdot 209^{2/3} = -0.221 \text{ barn}$$

The good agreement between the estimated and measured quadrupole moment is due to the fact that ^{209}Bi has one proton plus a double magic core. The double magic core of the nucleus is very rigid, therefore the additional proton causes little deformation from the spherical shape of the closed shells.

Exercise 2: Equipotential surfaces in the Rainwater-approximation

We saw that the axes of the ellipsoid can be described with the following equations:

$$A = R\left(1 - \frac{1}{3}\varepsilon\right) \qquad B = R\left(1 + \frac{2}{3}\varepsilon\right)$$

The equipotential surfaces will also be ellipsoids:

$$a = r_0\left(1 - \frac{1}{3}\varepsilon\right) \qquad b = r_0\left(1 + \frac{2}{3}\varepsilon\right)$$

The equation describing the ellipsoids is the following:

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1$$

$$\frac{x^2 + y^2}{\left(1 - \frac{1}{3}\varepsilon\right)^2} + \frac{z^2}{\left(1 + \frac{2}{3}\varepsilon\right)^2} = r_0^2$$

Rearranging the equation:

$$r_0^2 \cong (x^2 + y^2)\left(1 + \frac{2}{3}\varepsilon\right) + z^2\left(1 - \frac{4}{3}\varepsilon\right) = (x^2 + y^2 + z^2) + \frac{2}{3}\varepsilon(x^2 + y^2) - \frac{4}{3}\varepsilon z^2$$

where we used that $\frac{1}{(1+x)^2} \cong (1 - 2x)$ (from Taylor expansion).

$$r^2(\vartheta) = x^2 + y^2 + z^2 = r_0^2 - (x^2 + y^2 + z^2)\frac{2}{3}\varepsilon + 2z^2\varepsilon = r_0^2\left[1 + \frac{2}{3}\varepsilon\left(3\frac{z^2}{r_0^2} - \frac{x^2 + y^2 + z^2}{r_0^2}\right)\right]$$

we will also use the following:

$$\frac{z^2}{r_0^2} = \cos^2\vartheta \qquad \frac{x^2 + y^2 + z^2}{r_0^2} \approx 1$$

substituting back these equations we finally get:

$$r^2(\vartheta) = r_0^2\left[1 + \frac{2}{3}\varepsilon(3\cos^2\vartheta - 1)\right]$$

$$r(\vartheta) = r_0\left[1 + \frac{1}{3}\varepsilon(3\cos^2\vartheta - 1)\right]$$

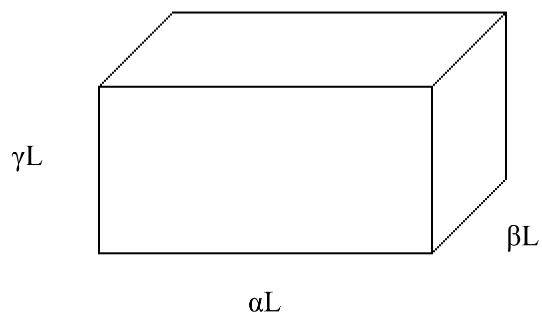
(where we also used an approximation from the Taylor-expansion)

Exercise 3: Cuboid shaped nucleus

Let us imagine a cuboid shaped nucleus with αL, βL and γL sidelengths. Assuming that the L^3 volume of the nucleus is preserved, what will be the shape of the nucleus if a) there are only closed shells, and b) there are nucleons in an open shell?

Solution:

The cuboid shaped nucleus corresponds to a system of fermions enclosed in a box:



where the $V(x,y,z)$ potential is 0 if $0<x<\alpha L$, $0<y<\beta L$, $0<z<\gamma L$, and ∞ elsewhere. The Schrödinger-equation will have the following form:

$$-\frac{\hbar^2}{2m}\Delta\varphi(x,y,z) = E\varphi(x,y,z)$$

with the following boundary conditions:

$$\varphi(0,y,z) = \varphi(\alpha L,y,z) = \varphi(x,0,z) = \varphi(x,\beta L,z) = \varphi(x,y,0) = \varphi(x,y,\gamma L) = 0$$

we know that the wavefunction will have the following form:

$$\varphi(x,y,z) = \sqrt{\frac{8}{V}}\sin(k_x x)\sin(k_y y)\sin(k_z z)$$

where V is the volume of the cuboid and

$$k_x = \frac{\pi}{\alpha L}n_x, \quad n_x = 1, 2, \ldots$$

$$k_y = \frac{\pi}{\beta L}n_y, \quad n_y = 1, 2, \ldots$$

$$k_z = \frac{\pi}{\gamma L} n_z, \quad n_z = 1, 2, ...$$

The energy of a nucleon with $n_{x,i}$, $n_{y,i}$ and $n_{z,i}$ quantum numbers can be calculated from the wavenumber:

$$E_i = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{\pi^2 n_{x,i}^2}{\alpha^2 L^2} + \frac{\pi^2 n_{y,i}^2}{\beta^2 L^2} + \frac{\pi^2 n_{z,i}^2}{\gamma^2 L^2}\right) = \frac{\hbar^2 \pi^2}{2mL^2}\left(\frac{1}{\alpha^2} n_{x,i}^2 + \frac{1}{\beta^2} n_{y,i}^2 + \frac{1}{\gamma^2} n_{z,i}^2\right)$$

And the total energy of the nucleons is the sum of the individual energies:

$$E = \sum_i E_i = \sum_i \frac{\hbar^2 \pi^2}{2mL^2}\left(\frac{1}{\alpha^2} n_{x,i}^2 + \frac{1}{\beta^2} n_{y,i}^2 + \frac{1}{\gamma^2} n_{z,i}^2\right)$$

We assumed that the volume is preserved, therefore:

$$\alpha\beta\gamma = 1 \quad \Rightarrow \quad \gamma = \frac{1}{\alpha\beta}$$

$$E = \sum_i \frac{\hbar^2 \pi^2}{2mL^2}\left(\frac{1}{\alpha^2} n_{x,i}^2 + \frac{1}{\beta^2} n_{y,i}^2 + \alpha^2\beta^2 n_{z,i}^2\right)$$

a) Let us first consider two nucleons in the box. According to the Pauli-principle both of them can be in the $n_x = n_y = n_z = 1$ state. The total energy of the system:

$$E = \frac{\hbar^2 \pi^2}{2mL^2} \cdot 2\left(\frac{1^2}{\alpha^2} + \frac{1^2}{\beta^2} + \alpha^2\beta^2 \cdot 1^2\right) = \frac{\hbar^2 \pi^2}{mL^2}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \alpha^2\beta^2\right)$$

In the equilibrium the energy is minimal, which means that the derivatives with α and β are zero:

$$\frac{\partial E}{\partial \alpha} = \frac{\hbar^2 \pi^2}{mL^2}\left(-\frac{2}{\alpha^3} + 2\alpha\beta^2\right) = 0$$

$$\frac{\partial E}{\partial \beta} = \frac{\hbar^2 \pi^2}{mL^2}\left(-\frac{2}{\beta^3} + 2\alpha^2\beta\right) = 0$$

From this we get two equations for α and β:

$$\alpha^4 = \frac{1}{\beta^2}, \qquad \beta^4 = \frac{1}{\alpha^2}$$

from which the solution shows a symmetrical nucleus for the closed shell: $\alpha = \beta = \gamma = 1$.

b) Let us consider an example where we put three protons or neutrons in the box. According to the Pauli-principle, two of them will be in the $n_x = n_y = n_z = 1$ state, while the third one will be in either

$$n_x = 1, \quad n_y = 1, \quad n_z = 2$$

$$n_x = 1, \quad n_y = 2, \quad n_z = 1$$

$$n_x = 2, \quad n_y = 1, \quad n_z = 1$$

These three states can be obtained from each other by rotating our coordinate system, therefore we only need to examine one of them:

$$E = \frac{\hbar^2 \pi^2}{2mL^2}\left[2\left(\frac{1^2}{\alpha^2} + \frac{1^2}{\beta^2} + \alpha^2 \beta^2 \cdot 1^2\right) + \left(\frac{1^2}{\alpha^2} + \frac{1^2}{\beta^2} + \alpha^2 \beta^2 \cdot 2^2\right)\right] = \frac{\hbar^2 \pi^2}{2mL^2}\left(\frac{3}{\alpha^2} + \frac{3}{\beta^2} + 6 \cdot \alpha^2 \beta^2\right)$$

In the equilibrium the energy is minimal, which means that the derivatives with α and β are zero:

$$\frac{\partial E}{\partial \alpha} = \frac{\hbar^2 \pi^2}{2mL^2}\left(-\frac{6}{\alpha^3} + 12 \cdot \alpha \beta^2\right) = 0$$

$$\frac{\partial E}{\partial \beta} = \frac{\hbar^2 \pi^2}{2mL^2}\left(-\frac{6}{\beta^3} + 12 \cdot \alpha^2 \beta\right) = 0$$

From this we get two equations for α and β:

$$\alpha^4 = \frac{1}{2\beta^2}, \qquad \beta^4 = \frac{1}{2\alpha^2} = \frac{1}{2\left(\frac{1}{2\beta^2}\right)^{1/2}} = 2^{-1/2} \cdot \beta$$

By solving the equations and using the volume preservation we get:

$$\alpha = \beta = 2^{-1/6}$$

$$\gamma = 2^{1/3}$$

We got that the equilibrium state of the nucleus is deformed and not a cube, because the third nucleon deformes the symmetrical core.