

# Nuclear Physics

## Practice 9

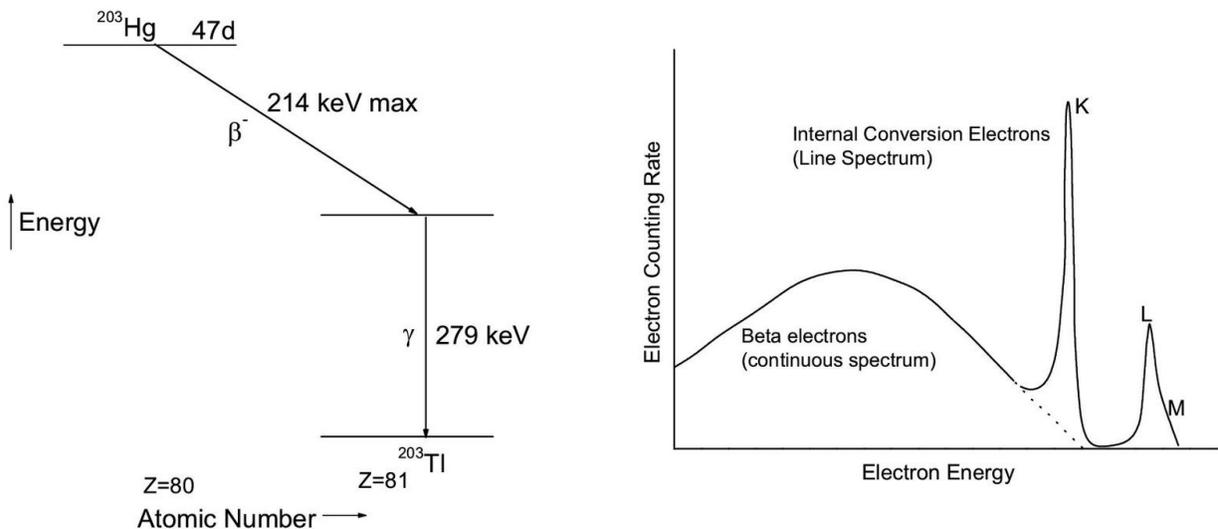
### Exercise 1: Internal conversion

Internal conversion (IC) is an electromagnetic process which competes with the  $\gamma$ -emission. In this case, however, the multipole field of the nucleus interacts with the atomic electrons, and instead of emitting a  $\gamma$ -photon, one of the electrons is emitted from the atom. Note that in contrast with the  $\beta$ -decay, the electron is not created in the decay process but previously exists in an atomic orbit. The  $\Delta E$  energy release of the nucleus will be transferred to the electron as kinetic energy, decreased by the binding energy needed to free the electron from its atomic shell:

$$T_e = \Delta E - B$$

It is obvious from the above formula that the internal conversion will have a threshold energy depending on the atomic shell (K, L, M, etc.) from which the electron is emitted. The electron leaves a vacancy on the corresponding shell which also results in the appearance of low-energy X-rays in the  $\gamma$ -spectrum (because electrons from outer shells will fill the vacancy).

Since  $\beta$ -decays often leave the daughter nucleus in an excited state, therefore IC electrons can appear in the otherwise continuous  $\beta$ -spectrum. Let us consider for example the decay scheme and  $\beta$ -spectrum of  $^{203}\text{Hg}$ :



The  $^{203}\text{Tl}$  daughter nucleus is in an excited state with 279.19 keV energy difference between the final state of the  $\beta$ -decay and the ground state. If we examine the  $\beta$ -spectrum of  $^{203}\text{Hg}$ , we can clearly see the discrete peaks of the conversion electrons. The electron energies correspond well with the binding energies of the atomic shells:

$$B(K) = 85.529 \text{ keV} \quad T_e(K) = 193.661 \text{ keV}$$

$$B(L_I) = 15.347 \text{ keV} \quad T_e(L_I) = 263.843 \text{ keV}$$

With high resolution even the different  $l$  orbital quantum numbers can be distinguished. The I index of the L shell in the second example means that the conversion electron originates from the  $2s_{1/2}$  atomic orbit. In some cases internal conversion is heavily favoured over  $\gamma$ -emission. As a general rule it is necessary to correct the total half-life of a particular level with the IC probability:

$$\lambda_{\text{tot}} = \lambda_\gamma + \lambda_{\text{IC}}$$

For later discussed reasons, it is convenient to define the internal conversion coefficient  $\alpha$  as the following:

$$\alpha = \frac{\lambda_{\text{IC}}}{\lambda_\gamma}$$

The total decay probability then becomes:

$$\lambda_{\text{tot}} = (1 + \alpha)\lambda_\gamma$$

We let  $\alpha$  represent the total internal conversion coefficient and define partial coefficients representing the individual atomic shells:

$$\lambda_{\text{tot}} = \lambda_\gamma + \lambda_K + \lambda_L + \dots = (1 + \alpha_K + \alpha_L + \dots)\lambda_\gamma$$

The reason behind introducing the internal conversion coefficient the previous way is that since both decay modes are electromagnetic processes, they will have similar matrix elements in Fermi's golden rule, except that the initial and final wavefunctions of the internal conversion will also contain the electron wavefunctions:

$$\Psi_i = \psi_{i,N} \varphi_{i,e}$$

$$\Psi_f = \psi_{f,N} \exp(-ik\underline{r}_e)$$

where we substituted the free electron wavefunction into the second expression. To a very good approximation, the atomic wavefunctions vary very little over the nucleus, and we can replace  $\varphi_{i,e}(\underline{r}_e)$  with  $\varphi_{i,e}(0)$ . The nuclear wavefunctions, as well as the multipole operator governing the transition are the same for  $\gamma$ -emission and internal conversion, therefore the nuclear part of the matrix elements are the same for both processes. Moreover, since  $k\underline{r}_e \ll 1$ , we can approximate  $\exp(ik\underline{r}_e) \approx 1$ . The decay probabilities over unit time are proportional to the matrix elements:

$$\lambda = \frac{2\pi}{\hbar} |\langle f | \hat{H} | i \rangle|^2 \rho$$

$$\frac{\lambda_{\text{IC}}}{\lambda_\gamma} \propto \frac{\left| \int \psi_{f,N}^*(\underline{r}) \exp(ik\underline{r}_e) \hat{H} \psi_{i,N}(\underline{r}) \varphi_{i,e}(\underline{r}) d^3r \right|^2}{\left| \int \psi_{f,N}^*(\underline{r}) \hat{H} \psi_{i,N}(\underline{r}) d^3r \right|^2} \approx |\varphi_{i,e}(0)|^2$$

therefore the internal conversion coefficient is *independent* of the nuclear properties (with the above approximations!). A nonrelativistic calculation gives the following instructive results on the coefficients:

$$\alpha(EI) \cong \frac{Z^3}{n^3} \left( \frac{l}{l+1} \right) \left( \frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{\hbar c} \right)^4 \left( \frac{2m_e c^2}{E} \right)^{l+5/2}$$

$$\alpha(MI) \cong \frac{Z^3}{n^3} \left( \frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{\hbar c} \right)^4 \left( \frac{2m_e c^2}{E} \right)^{l+3/2}$$

where  $Z$  is the atomic number,  $l$  describes the multipolarity of the EM transition,  $n$  is the atomic shell and  $E$  is the transition energy. From these the following qualitative considerations can be concluded:

- the value of  $\alpha$  is bigger for nuclei with higher atomic number (e.g.  $^{22}\text{Ne}$   $\alpha_K=6.8 \cdot 10^{-6}$ ,  $^{182}\text{W}$   $\alpha_K=2.5 \cdot 10^{-3}$ )
- internal conversion gets less probable as the  $E$  transition energy increases.
- the probability of IC increases with multipolarity (e.g.  $^{99}\text{Tc}$  141 keV M1:  $\alpha_K=0.10$ , 143 keV M4:  $\alpha_K=30$ )
- inner electrons are emitted with higher probability (e.g.  $\alpha_K/\alpha_L=8$ )

The expectation is that internal conversion is the most probable for low-energy, K-shell, high multipolarity transitions of heavy nuclei. Another important case is the  $0 \rightarrow 0$  transition:

$$|0-0| < l < |0-0| \Rightarrow l=0$$

The  $\gamma$ -emission is therefore forbidden in these cases, the nucleus can only release its excitation energy with internal conversion.

## Exercise 2: Spin-orbit interaction

E.P. Wigner, M. Goeppert-Mayer, H. Jensen: spin-orbit interaction is also present in the nucleus, analogous to the interaction between the spin and orbital momentum of atomic electrons

Classical picture: The electron in the H atom is orbiting around the proton with  $v_e$  velocity. For infinitesimal displacements the electron sees that the proton is orbiting around it with  $-v_e$  velocity. From the point of view of the electron the proton creates  $B$  magnetic field.

According to Biot-Savart law (in SI system):

$$\Delta \underline{B} = \frac{\mu_0}{4\pi} \frac{I \Delta \underline{l} \times \underline{r}}{r^3}$$

The current times displacement vector can be written as:

$$I \Delta \underline{l} = -e \underline{v}_e$$

which gives us

$$\Delta \underline{B} = -\frac{\mu_0}{4\pi} \frac{e \underline{v}_e \times \underline{r}}{r^3} = \frac{\mu_0 e}{4\pi m} \cdot \frac{\underline{r} \times \underline{p}}{r^3} = \frac{\mu_0 e \hbar}{4\pi m} \cdot \frac{\underline{L}}{r^3}$$

where we used the definitions of the momentum and the angular momentum:

$$\underline{p} = m \underline{v}$$

$$\underline{L} = \frac{1}{\hbar} \underline{r} \times \underline{p}$$

The interaction energy between the spin of the electron and the magnetic field created by the proton:

$$E = -\underline{\mu}_s \cdot \underline{B}$$

where  $\underline{\mu}_s$  is expressed with the spin:

$$\underline{\mu}_s = \frac{e \hbar}{m} \underline{S}$$

Let us substitute the magnetic field into the interaction operator:

$$\hat{U} = -\frac{e \hbar}{m} \underline{S} \cdot \frac{\mu_0 e \hbar}{4\pi m} \cdot \frac{\underline{L}}{r^3} = -\frac{\mu_0}{4\pi} \left( \frac{e \hbar}{m} \right)^2 \cdot \frac{1}{r^3} \cdot \underline{L} \underline{S} = -\left( \frac{\hbar}{mc} \right)^2 \cdot \frac{1}{r} \cdot \frac{dV_C}{dr} \cdot \underline{L} \underline{S}$$

given that  $c = 1/\sqrt{\mu_0 \epsilon_0}$  and the space derivative of the Coulomb-potential is:

$$\frac{dV_C}{dr} = \frac{1}{4\pi \epsilon_0} \frac{e^2}{r^2}$$

Suppose that the spin-orbit interaction is also present in the nucleus, and the interaction operator can also be originated from the  $V(r)$  potential of the nuclear force with a  $\lambda$  coupling constant:

$$\hat{U}_{L,S} = -\lambda \left( \frac{\hbar}{mc} \right)^2 \cdot \frac{1}{r} \cdot \frac{dV}{dr} \cdot \underline{L} \underline{S}$$

where the nuclear potential is the following:

$$V(r) = V_0 \left( \frac{r^2}{R^2} - 1 \right), \text{ if } r < R$$

We shall investigate the effect of the spin-orbit interaction using perturbation theory:

$$\hat{H} \rightarrow \hat{H} + \hat{U}_{L,S}$$

the interaction operator using the above nuclear potential:

$$\hat{U}_{LS} = -\lambda \left( \frac{\hbar}{mc} \right)^2 \cdot \frac{2V_0}{R^2} \cdot \underline{L} \underline{S} = -V_{0,LS} \cdot \underline{L} \underline{S}$$

The first order term of the energy change due to spin-orbit interaction can be written as:

$$E_{LS} = \langle \varphi_{j,m_j,l,s} | \hat{U}_{LS} | \varphi_{j,m_j,l,s} \rangle = -V_{0,LS} \langle \varphi_{j,m_j,l,s} | \underline{L} \underline{S} | \varphi_{j,m_j,l,s} \rangle$$

To calculate the energy change, we must determine the value of  $LS$ . We will use the following trick:

$$\underline{J} = \underline{L} + \underline{S}$$

$$J^2 = L^2 + S^2 + 2\underline{L} \underline{S} \rightarrow \underline{L} \underline{S} = \frac{1}{2} (J^2 - L^2 - S^2)$$

Let us substitute this expression into the formula of  $E_{LS}$ :

$$E_{LS} = -V_{0,LS} \langle \varphi_{j,m_j,l,s} | \frac{1}{2} (J^2 - L^2 - S^2) | \varphi_{j,m_j,l,s} \rangle = -\frac{1}{2} V_{0,LS} \left[ j(j+1) - l(l+1) - \frac{1}{2} \left( \frac{1}{2} + 1 \right) \right] \hbar^2$$

where  $j$  is the sum of the orbital momentum and the spin:  $j = l \pm \frac{1}{2}$

1) if  $j = l + \frac{1}{2}$ :

$$\left( l + \frac{1}{2} \right) \left( l + \frac{3}{2} \right) - l(l+1) - \frac{3}{4} = l^2 + 2l + \frac{3}{4} - l^2 - l - \frac{3}{4} = l$$

$E_{LS} < 0 \rightarrow$  the energy level moves downwards

2) if  $j = l - \frac{1}{2}$ :

$$\left( l - \frac{1}{2} \right) \left( l + \frac{1}{2} \right) - l(l+1) - \frac{3}{4} = l^2 - \frac{1}{4} - l^2 - l - \frac{3}{4} = -l - 1$$

$E_{LS} > 0 \rightarrow$  the energy level moves upwards