Nuclear Physics

Practice 6

Exercise 1: Multipole expansion of the wave equation

The nucleus can be in different states described by different parities and angular momenta:

- ground state: I_0, π_0
- excited states: I_N, π_N

Important rule: the emitted γ-photon carries the conservation of angular momentum, parity and energy.

→ The transition can be determined by detecting the γ-photon.

The place of the detection is far from the source, therefore we can apply the Maxwell-equations in vacuum:

$$\nabla \underline{B} = 0$$

$$\nabla \underline{E} = 0$$

$$\nabla \times \underline{B} = \mu_0 \varepsilon_0 \dot{E} = \frac{1}{c^2} \underline{\dot{E}}$$

$$\nabla \times \underline{E} = -\underline{\dot{B}}$$

We will apply a special treatment, the multipole expansion, in order to be able to describe the angular momentum and parity of the radiation. Let us consider a harmonic field approximation:

$$\underline{E}(\underline{r}, t) = \underline{E}(\underline{r}) \exp(-i\omega t)$$

$$\underline{B}(\underline{r}, t) = B(\underline{r}) \exp(-i\omega t)$$

From the Maxwell equation, we can write (with the wavenumber $k=\omega/c$):

$$\nabla \times \underline{E} = i\omega \underline{B} \rightarrow \underline{B} = -\frac{i}{ck} \nabla \times \underline{E}$$

$$\nabla \times \underline{B} = -\frac{i\omega}{c^2} \underline{E} \rightarrow \underline{E} = \frac{ic}{k} \nabla \times \underline{B}$$

We will use the following identities:

$$\nabla \times (\nabla \times \underline{E}) = \text{grad div}\, \underline{E} - \Delta \underline{E}$$

$$\nabla \times (\nabla \times \underline{B}) = \text{grad div}\, \underline{B} - \Delta \underline{B}$$

From the above equations we get:

$$\nabla \times (\nabla \times \underline{E}) = \nabla \times (i\omega \underline{B}) = i\omega \nabla \times \underline{B} = i\omega \left(-\frac{i\omega}{c^2} \underline{E} \right) = k^2 \underline{E} = 0 - \Delta \underline{E} \quad \Rightarrow \quad (\Delta + k^2) \underline{E} = 0$$

$$\nabla\times(\nabla\times\underline{B})=\nabla\times\left(-\frac{i\omega}{c^2}\underline{E}\right)=-\frac{i\omega}{c^2}\nabla\times\underline{E}=-\frac{i\omega}{c^2}(i\omega\underline{B})=k^2\underline{B}=0-\Delta\underline{B}\quad\Rightarrow\quad(\Delta+k^2)\underline{B}=0$$

These are the Helmholtz-equations, which give 2x3=6 equations for the different components, and which are coupled through the divergence equations. Since their solution is still complicated, we will use another approach. If div($\underline{A}$)=0 for an $\underline{A}$ vector field, then:

$$\underline{r}\Delta\underline{A}=\Delta(\underline{r}\underline{A})$$

Therefore the Helmholtz-equations will also be true for these scalar products:

$$(\Delta+k^2)\underline{r}\underline{E}=0$$

$$(\Delta+k^2)\underline{r}\underline{B}=0$$

Theorem:

$$(\Delta+k^2)\Phi(\underline{r})=0$$

$$\Rightarrow\quad\Phi(\underline{r})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{lm}f_l(kr)Y_l^m(\underline{\Omega})$$

where $f_l(kr)$ are the spherical Hankel-functions and $Y_l^m(\underline{\Omega})$ are the spherical harmonics.

Using the above theorem, the $\underline{rE}$ and $\underline{rB}$ scalar products can be also expanded:

$$\underline{r}\underline{E}=\sum_{l,m}a_{lm}f_l(kr)Y_l^m(\underline{\Omega})$$

$$\underline{r}\underline{B}=\sum_{l,m}b_{lm}f_l(kr)Y_l^m(\underline{\Omega})$$

We can give $\underline{E}$ and $\underline{B}$ with the linear combinations of the transverse electric (TE) and transverse magnetic (TM) modes:

- TE mode: $\underline{r}\underline{E}=0$ (magnetic transition) → $B^{(m)}$, $E^{(m)}$
- TM mode: $\underline{r}\underline{B}=0$ (electric transition) → $B^{(e)}$, $E^{(e)}$

The solution can be obtained with the following transformations, using $\underline{L}=-i\hbar(\underline{r}\times\nabla)$:

1) TE mode:

$$\underline{r}\underline{B}_{l,m}^{(m)}=\underline{r}\left(-\frac{i}{ck}\nabla\times\underline{E}_{l,m}^{(m)}\right)=-\frac{i}{ck}\underline{r}(\nabla\times\underline{E}_{l,m}^{(m)})=-\frac{i}{ck}(\underline{r}\times\nabla)\underline{E}_{l,m}^{(m)}=-\frac{i}{ck}\cdot\left(-\frac{1}{i\hbar}\right)\underline{L}\underline{E}_{l,m}^{(m)}=\frac{1}{ck\hbar}\underline{L}\underline{E}_{l,m}^{(m)}$$

$$\Rightarrow\quad\frac{1}{ck\hbar}\underline{L}\underline{E}_{l,m}^{(m)}=a_{lm}f_l(kr)Y_l^m(\underline{\Omega})$$

applying the $\underline{L}$ angular momentum operator on both sides we get:

$$\frac{1}{ck\hbar}\underline{L}^2\underline{E}_{l,m}^{(m)}=a_{lm}f_l(kr)\underline{L}Y_l^m(\underline{\Omega})$$

Assuming that $\underline{E}_{l,m}^{(m)}$ also contains the spherical harmonics, we are able to use the eigenvalue equation of the $\underline{L}^2$ operator:

$$\underline{L}^2 Y_l^m(\underline{\Omega}) = \hbar^2 l(l+1) Y_l^m(\underline{\Omega})$$

$$\underline{E}_{l,m}^{(m)} = \frac{ck}{\hbar l(l+1)} a_{lm} f_l(kr) \underline{L} Y_l^m(\underline{\Omega}) = \tilde{a}_{lm} f_l(kr) \underline{L} Y_l^m(\underline{\Omega})$$

Now we can express the TE mode of $\underline{B}_{l,m}^{(m)}$ using the rotation:

$$\underline{B}_{l,m}^{(m)} = -\frac{i}{ck} \tilde{a}_{lm} \nabla \times \left[f_l(kr) \underline{L} Y_l^m(\underline{\Omega}) \right]$$

2) TM mode:

A similar derivation leads us to the following results:

$$\underline{B}_{l,m}^{(e)} = \tilde{b}_{lm} f_l(kr) \underline{L} Y_l^m(\underline{\Omega})$$

$$\underline{E}_{l,m}^{(e)} = \frac{ic}{k} \tilde{b}_{lm} \nabla \times \left[f_l(kr) \underline{L} Y_l^m(\underline{\Omega}) \right]$$

The total electromagnetic field is the sum of the TE and TM modes:

$$\underline{E} = \sum_{l,m} \left[\tilde{a}_{lm} f_l(kr) \underline{L} Y_l^m(\underline{\Omega}) + \frac{ic}{k} \tilde{b}_{lm} \nabla \times (f_l(kr) \underline{L} Y_l^m(\underline{\Omega})) \right]$$

$$\underline{B} = \sum_{l,m} \left[\tilde{b}_{lm} f_l(kr) \underline{L} Y_l^m(\underline{\Omega}) - \frac{i}{ck} \tilde{a}_{lm} \nabla \times (f_l(kr) \underline{L} Y_l^m(\underline{\Omega})) \right]$$

These are true in the case where there is no source. Problem: the nucleus.

→ The presence of the nucleus will determine the $\tilde{a}_{lm}$, $\tilde{b}_{lm}$ coefficients (complicated quantum mechanical calculation).

The angular distribution of the power can be calculated with the Poynting-vector:

$$P_{l,m}^{(m)}(\underline{\Omega}) \sim |\tilde{a}_{lm}|^2 |\underline{L} Y_l^m(\underline{\Omega})|^2$$

$$P_{l,m}^{(e)}(\underline{\Omega}) \sim |\tilde{b}_{lm}|^2 |\underline{L} Y_l^m(\underline{\Omega})|^2$$

The total emitted power:

$$P_{l,m}^{(m)} = \int_{4\pi} P_{l,m}^{(m)}(\underline{\Omega}) d\underline{\Omega} \sim |\tilde{a}_{lm}|^2$$

$$P_{l,m}^{(e)} = \int_{4\pi} P_{l,m}^{(e)}(\underline{\Omega}) d\underline{\Omega} \sim |\tilde{b}_{lm}|^2$$

where we used the following:

$$\int_{4\pi} | \underline{L} Y_l^m(\underline{\Omega}) | \, d\underline{\Omega} = \int_{4\pi} Y_l^{m*}(\underline{\Omega}) \underline{L}^2 Y_l^m(\underline{\Omega}) d\underline{\Omega} = \hbar^2 l(l+1)$$

The transition probability per unit time can be calculated by dividing with the photon energy:

$$W_{21} = \frac{P_{l,m}^{(e,m)}}{\hbar\omega}$$

The same results arise from Fermi's golden rule.

Parity:

$$I_2^{\pi 2} \rightarrow I_1^{\pi 1} + (l,m)$$

From the angular momentum conservation we can tell which transitions are possible from the initial state:

$$| I_2 - I_1 | \le l \le | I_2 + I_1 |$$

$$m = M_2 - M_1$$

The parity can be determined from the golden rule:

$$\lambda_{fi} \sim | M_{fi} |^2 = \left| < \mathrm{f} \mid H_I \mid \mathrm{i} > \right|^2$$

The electromagnetic field can be described with the $\underline{A}$ vector potential and Φ scalar potential:

$$\underline{B} = \nabla \times \underline{A}$$

$$\underline{E} = -\nabla \Phi - \frac{\partial \underline{A}}{\partial t}$$

Since Φ decreases quickly, $\underline{A}$ will determine the interaction Hamilton-operator:

$$H_I \sim \underline{p}\underline{A}$$

TE mode (magnetic transition): the parity of $\underline{B}$ is $(-1)^{l+1} \rightarrow \pi(\underline{pA}) = (-1)^{l+1}$

TM mode (electric transition): the parity of $\underline{B}$ is $(-1)^{l} \rightarrow \pi(\underline{pA}) = (-1)^{l}$

Example:

$$3^- \rightarrow 1^+$$

$$| 3-1 | \le l \le | 3+1 |$$

$$l = 2,3,4$$

The parity of the transition is (-1), therefore E3, M2 and M4 are possible transitions. Note that in some of the literature TE is called electric transition and TM is called magnetic transition!