Nuclear Physics

Practice 3

Exercise 1: Linear Accelerator (LINAC)

At the Stanford Linear Accelerator (SLAC) electrons are accelerated along the two mile accelerator to an energy of 6 GeV. We have seen that the diffraction of the particles depends on their wavelengths and the higher momentum a particle has the shorter its wavelength is therefore the finer are the details that can be observed. How fine resolution can be obtained with these electrons?

$$m_e = 9.109382 \cdot 10^{-31} \text{ kg}$$

$$h = 6.62607 \cdot 10^{-34} \ \frac{\text{m}^2\text{kg}}{\text{s}}$$

$$e = 1.602177 \cdot 10^{-19} \text{ C}$$

Solution:

Let us utilize the formula for the kinetic energy to calculate the velocity of the electrons:

$$E_{kin} = \frac{1}{2} m v^2$$

from this we get the following velocity:

$$v = \sqrt{\frac{2E_{kin}}{m}} = 4.5910 \cdot 10^{10} \ \frac{\text{m}}{\text{s}}$$

we see that $v > c$ which is impossible. Solution: we have to calculate with relativistic formulas ($E_{kin} >> m_0 c^2$ already suggested this).

$$E = mc^2$$

$$m = \frac{E}{c^2}$$

and the momentum can be expressed as:

$$p = mv = \frac{E}{c^2} v$$

since at this high energies $v \approx c$ we can use the following approximation:

$$p = mv = \frac{E}{c^2} v \approx \frac{E}{c^2} c = \frac{E}{c} = 3.2043 \cdot 10^{-18} \ \frac{\text{kgm}}{\text{s}}$$

According to the de Broglie formula the wavelength of the electrons will be:

$$\lambda = \frac{h}{p} = 2.0678 \cdot 10^{-16} \text{ m}$$

These electrons can „see inside” a nucleon!

Exercise 2: Random walk (Drunken sailor)

Let us consider a one-dimensional symmetric random walk, and let us denote the displacement after each step with s_i. For a symmetric random walk the probabilities of left and right steps are equal:

$$s_i = \begin{cases} +d & \text{with 50\% probability} \\ -d & \text{with 50\% probability} \end{cases}$$

After N steps in the random walk, the position x of the walker is the sum of the s_i displacements:

$$x = \sum_{i=1}^{N} s_i$$

and the position squared is:

$$x^2 = \left(\sum_{i=1}^{N} s_i \right)^2$$

It is immediately obvious that for a symmetric walk the estimation value of the position will always be zero:

$$< x >= 0$$

Of course this does not mean that the particle is always at zero, it just means that the probability distribution of finding the particle somewhere is centered at zero, but the probability distribution gets wider with increasing number of N. Let us consider the estimation value of the squared distance which we can rearrange as the following:

$$x^2 = \left(\sum_{i=1}^{N} s_i \right)^2 = \sum_{i=1}^{N} s_i \sum_{j=1}^{N} s_j = \sum_{i=1}^{N} s_i^2 + \sum_{i \neq j} s_i s_j$$

where the multiplication for an $s_i s_j$ pair will be:

$$s_i s_j = \begin{cases} +d^2 & \text{with 50\% probability} \\ -d^2 & \text{with 50\% probability} \end{cases}$$

so on average the sum over $s_i s_j$ will be zero. On the other hand:

$$s_i^2 = d^2$$

independently of whether s_i is $+d$ of $-d$. Therefore the average squared displacement of N steps will be:

$$< x^2 >= d^2 N$$

Finally from the definition of the deviation we get: $\sigma_x = \sqrt{< x^2 > - < x >^2} = d\sqrt{N}$

Exercise 3: Decay chains

Consider the following decay chain:

$$X_1 \rightarrow X_2 \rightarrow X_3$$

where the third isotope of the chain, X_3 is stable. Let $N_1(t)$, $N_2(t)$ and $N_3(t)$ be the corresponding number densities for the three nuclides of which $N_2(0)$ and $N_3(0)$ are zeros.

The first order differential equation system that describes the decay chain is the following:

$$\frac{dN_1(t)}{dt} = -\lambda_1 N_1(t)$$

$$\frac{dN_2(t)}{dt} = \lambda_1 N_1(t) - \lambda_2 N_2(t)$$

$$\frac{dN_3(t)}{dt} = \lambda_2 N_2(t)$$

The first one is a separable differential equation whose solution has been already shown in the lecture:

$$N_1(t) = N_{1,0} \exp(-\lambda_1 t)$$

Substituting this time dependency into the second equation we get:

$$\frac{dN_2(t)}{dt} = \lambda_1 N_{1,0} \exp(-\lambda_1 t) - \lambda_2 N_2(t)$$

This is a first order linear inhomogeneous differential equation system which for example can be solved with the method known as the variation of constants. First we consider the homogeneous equation:

$$\frac{dN_2(t)}{dt} = -\lambda_2 N_2(t)$$

The solution of the homogeneous equation is already known:

$$N_{2,h}(t) = N_{2,0} \exp(-\lambda_2 t)$$

The variation of constants means that we suppose that this $N_{2,0}$ is not constant but also time dependent:

$$N_2(t) = N_{2,0}(t) \exp(-\lambda_2 t)$$

then we substitute this expression into the second equation:

$$\frac{d}{dt}\left[N_{2,0}(t) \exp(-\lambda_2 t)\right] = \lambda_1 N_{1,0} \exp(-\lambda_1 t) - \lambda_2 N_{2,0}(t) \exp(-\lambda_2 t)$$

$$\exp(-\lambda_2 t)\cdot\frac{dN_{2,0}(t)}{dt}-\lambda_2\exp(\lambda_2 t)N_{2,0}(t)=\lambda_1 N_{1,0}\exp(-\lambda_1 t)-\lambda_2 N_{2,0}(t)\exp(-\lambda_2 t)$$

simplifying with the second expression on each side of the equation we get:

$$\exp(-\lambda_2 t)\cdot\frac{dN_{2,0}(t)}{dt}=\lambda_1 N_{1,0}\exp(-\lambda_1 t)$$

$$\frac{dN_{2,0}(t)}{dt}=\lambda_1 N_{1,0}\exp(-(\lambda_1-\lambda_2)t)$$

we integrate the expression to obtain $N_{2,0}(t)$:

$$N_{2,0}(t)=\int_0^t \lambda_1 N_{1,0}\exp(-(\lambda_1-\lambda_2)\tilde{t})d\tilde{t}=\lambda_1 N_{1,0}\left[-\frac{1}{\lambda_1-\lambda_2}\cdot\exp(-(\lambda_1-\lambda_2)t)\right]_0^t$$

$$N_{2,0}(t)=\frac{\lambda_1}{\lambda_1-\lambda_2}N_{1,0}\left[1-\exp(-(\lambda_1-\lambda_2)t)\right]$$

the total solution of $N_2(t)$ will be therefore:

$$N_2(t)=\frac{\lambda_1}{\lambda_1-\lambda_2}N_{1,0}\left[1-\exp(-(\lambda_1-\lambda_2)t)\right]\cdot\exp(-\lambda_2 t)$$

which we have already seen in the lecture.

Exercise 3+1: Cyclotron

We saw that the centripetal force is equal to the Lorentz force:

$$F_{cp} = \frac{mv^2}{r} = qvB$$

the frequency can be expressed in the following way:

$$\omega = \frac{v}{r} = \frac{q}{m} B = \text{const.}$$

Problem: if $v \to c$ then m is not constant.

Calculate the relative decrease in frequency at 10 MeV, 250 MeV and 1 GeV proton energies!

$$m_0 c^2 = 938.272046 \text{ MeV}$$

Solution:

The ratio of the original and changed frequencies is the following:

$$\frac{\omega}{\omega_0} = \frac{\frac{q}{m} B}{\frac{q}{m_0} B} = \frac{m_0}{m}$$

The relativistic energy formula:

$$E = E_0 + E_{kin} = mc^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\frac{\omega}{\omega_0} = \frac{E_0}{E} = \frac{E_0}{E_0 + E_{kin}}$$

1)

$$E_{kin} = 10\ MeV: \ \frac{\omega}{\omega_0} = 0.9895 \ (\approx 1\% \text{ decrease})$$

2)

$$E_{kin} = 250\ MeV: \ \frac{\omega}{\omega_0} = 0.7896 \ (\approx 21\% \text{ decrease})$$

3)

$$E_{kin} = 1\, GeV: \ \frac{\omega}{\omega_0} = 0.4841 \ (\approx 52\% \text{ decrease})$$