Nuclear Physics
(9^{th} lecture)

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The nuclear shell model (contd.)

Magic numbers: 2, 8, 20, 28, 50, 82, 126

We assumed spherical central potential

$$\psi_{n_r,l,m}(r,\vartheta,\varphi)=\frac{\phi_{n_r}(r)}{r}Y_l^m(\vartheta,\varphi)\rightarrow\begin{cases}n_r=0,1,2,3...\\ l=0,1,2,3...\\ -l\le m\le +l\end{cases}$$

n_r - radial quantum number
l - orbital quantum number
m - magnetic quantum number

1^{st} case: the infinite well potential

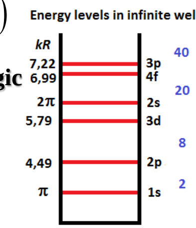
$$V(r)=\begin{cases}-V_0,\text{ if } r\le R,\\ \infty,\text{ if } r>R\end{cases}$$

$$k^2_{n_r,\ell}=\frac{2M}{\hbar^2}\left(V_0-E_{n_r,\ell}\right)$$

$$\phi_{n_r,\ell}(r)=j_l\left(k_{n_r,\ell}r\right)$$

kR		$2\cdot(2l+1)$	Sum	
π	$l=0$ (1s)	2	2	2
4,49	$l=1$ (2p)	6	8	8
5,79	$l=2$ (3d)	10	18	
2π	$l=0$ (2s)	2	20	20
6,99	$l=3$ (4f)	14	34	?
7,22	$l=1$ (3p)	6	40	

The first 3 magic numbers are okay, but the rest?



The nuclear shell model (contd.)

Magic numbers: 2, 8, 20, 28, 50, 82, 126

2^{nd} case: the harmonic oscillator potential

$$V(r)=-V_0+\frac{1}{2}m\omega^2r^2$$

Descartes			spherical				
n	π	$(n+1)(n+2)$	n_r	l		$2\cdot(2l+1)$	
0	+	2	0	0	1s	2	2 ☺
1	–	6	0	1	2p	6	8 ☺
2	+	12	1	0	2s	2	
			0	2	3d	10	20☺
3	–	20	1	1	3p	6	
			0	3	4f	14	40☹
4	+	30	2	0	3s	2	
			1	2	4d	10	
			0	4	5g	18	70☹

$$\boxed{\psi_n(-\mathbf{r})=(-1)^n\psi_n(\mathbf{r})}$$

$$\boxed{2g=(n+1)(n+2)}$$

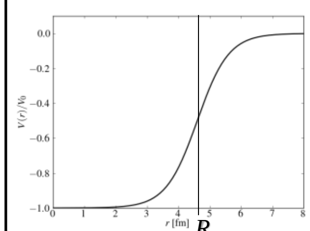
$$\boxed{n=2n_r+l}$$

The harmonic oscillator can describe only the first 3 magic numbers. Something more should be included!

The nuclear shell model (contd.)

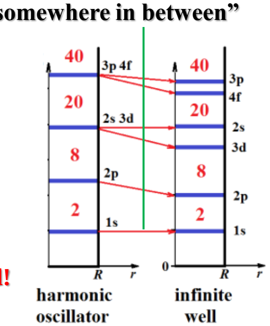
Magic numbers: 2, 8, 20, 28, 50, 82, 126

3^{rd} case: the Saxon-Woods potential



$$V(r)=-V_0\frac{1}{1+e^{\frac{r-R}{a}}}$$

Saxon-Woods is „somewhere in between”



Analytical solution of the Schrödinger equation is not possible. Numerical solutions show that the harmonic oscillator levels separate according to the l quantum number.

The magic numbers cannot be explained! Something else is needed!

The nuclear shell model (contd.)

E.P. Wigner, M. Goeppert-Mayer, H. Jensen (Nobel-prize 1963)

There should be a spin-orbit interaction term!

$$\left(-\frac{\hbar^2}{2m}\Delta + V(r) + V_{LS}(r)\cdot\mathbf{LS}\right)\psi(r,\vartheta,\varphi) = E\cdot\psi(r,\vartheta,\varphi)$$

The total angular momentum: $\mathbf{J} = \mathbf{L} + \mathbf{S}$

$J^2 = L^2 + S^2 + 2\cdot\mathbf{LS}$ **from where:** $\mathbf{LS} = \frac{1}{2}\left(J^2 - L^2 - S^2\right)$

The eigenvalue of the LS operator $= \frac{1}{2}\left(j(j+1) - l(l+1) - s(s+1)\right)$

Since $s = \frac{1}{2}$ **and** $l = j \pm \frac{1}{2}$

the eigenvalue of the LS operator $= \begin{cases} -\frac{1}{2}\left(j+\frac{3}{2}\right), \text{ if } j = l - \frac{1}{2} \\ +\frac{1}{2}\left(j-\frac{1}{2}\right), \text{ if } j = l + \frac{1}{2} \end{cases}$

Degeneration gets split!

If the $V_{LS}(r)$ potential is attractive, then $j = l+1/2$ state gets lower energy and the $j = l-1/2$ state gets higher energy.

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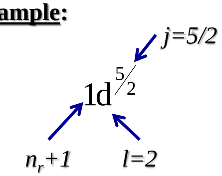
The nuclear shell model

Remember: $\psi_{n_r,l,m}(r,\vartheta,\varphi) = \frac{\phi_{n_r}(r)}{r} Y_l^m(\vartheta,\varphi) \to \begin{cases} n_r = 0,1,2,3\ldots \\ l = 0,1,2,3\ldots \\ -l \le m \le +l \end{cases}$

these are the 3 „natural" quantum numbers

In nuclear physics the notation of the states is NOT according to the *n* „main" quantum number like in atomic physics (since the main quantum number depends on the potential), but on the n_r „radial" quantum number

Example:



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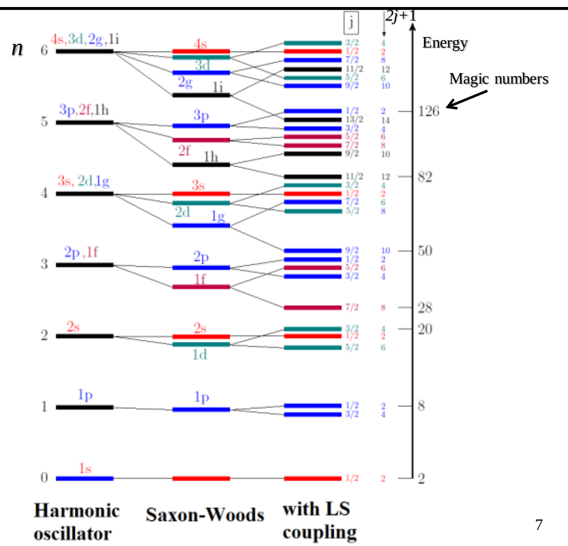

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The nuclear shell model

Excited states:

- **Simple case: even-even core + one particle (p or n) outside the core. The proton and neutron potentials are slightly different (Coulomb potential). Single-particle excitation energies reasonably well described.**
- **More complicated case: odd-odd nucleus → two particles (1n,1p) outside the core. The residual interaction between the unpaired neutron and proton should also be taken into consideration.**

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The nuclear shell model

Nuclear spin and parity (ground state):

- **Even-even nuclei:** $I = 0$, $\pi = +$ **(because of the pairing)**
- **Even-odd nuclei:** $I = j$ **(the angular momentum of the unpaired)**
  $\pi = (-1)^l$
- **Odd-odd nuclei: the coupling between the two unpaired nucleon should also be taken into account.**

Nucleus	No. of odd particles	Predicted configuration and (parity)	Observed spin & (parity)
^{11}B	5	p- hole, $p_{3/2}(-)$	3/2(-)
^{13}C	7	n- hole, $p_{1/2}(-)$	1/2(-)
^{15}N	7	p- hole, $p_{1/2}(-)$	1/2(-)
^{17}O	9	n- particle, $d_{5/2}(+)$	5/2(+)
^{39}K	19	p- hole, $d_{3/2}(+)$	3/2(+)
^{59}Co	27	p- hole, $f_{7/2}(-)$	7/2(-)
^{87}Sr	49	n- hole, $g_{9/2}(+)$	9/2(+)
^{207}Pb	125	n- hole, $p_{1/2}(-)$	1/2(-)

The nuclear shell model (contd.)

Nuclear magnetic moments:

Remember: In general: $\boxed{\boldsymbol{\mu} = g \cdot \mathbf{J} \cdot \mu_N}$ g **is the „gyromagnetic factor"**
Dipole magnetic moments can be originated from two sources:

- **„revolving" charge (e.g. protons on $l > 0$ orbits)** $\boldsymbol{\mu}_l = g_l \mathbf{l} \mu_N$
- **Nucleons have „intrinsic" magnetic moments:** $\boldsymbol{\mu}_s = g_s \mathbf{s} \mu_N$
  μ_s**(p) = 2,7928456** μ_N
  μ_s**(n) = –1,91304185** μ_N,
  where μ_N **is the „nuclear magneton".** $\mu_N = \frac{e \cdot \hbar}{2M_p} = 3{,}1525 \cdot 10^{-8}\ \frac{\text{eV}}{\text{T}}$

Problem: $\boldsymbol{\mu} = \boldsymbol{\mu}_l + \boldsymbol{\mu}_s = g_l \mathbf{l} + g_s \mathbf{s} \neq g_j \mathbf{j}$

Since in general $g_l \neq g_s$, **therefore μ and j are not parallel!**

Only the projection of μ on j can be measured!

$$\mu_{eff} = \frac{\boldsymbol{\mu}\mathbf{j}}{j} = \left(g_l \frac{\mathbf{lj}}{j} + g_s \frac{\mathbf{sj}}{j} \right) \mu_N$$

and the direction: $\boldsymbol{\mu}_{eff} = g_j \mathbf{j} \mu_N = \left(g_l \frac{\mathbf{lj}}{j} + g_s \frac{\mathbf{sj}}{j} \right) \cdot \frac{\mathbf{j}}{j} \mu_N$

Nuclear magnetic moments (contd.)

From the previous: $g_j = g_l \frac{\mathbf{lj}}{j^2} + g_s \frac{\mathbf{sj}}{j^2}$

The usual trick: from $\mathbf{j} = \mathbf{l} + \mathbf{s}$ **we get** $\mathbf{lj} = \frac{1}{2}\left(j^2 + l^2 - s^2\right)$

and $\mathbf{sj} = \frac{1}{2}\left(j^2 + s^2 - l^2\right)$

Using these $g_j = g_l \frac{j(j+1) + l(l+1) - s(s+1)}{2j(j+1)} + g_s \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)}$

Since $j = l \pm \frac{1}{2}$ **we get** $\boxed{g_j = \left(g_l \pm \frac{1}{2l+1}(g_s - g_l) \right)}$

and $\mu_{eff} = j\left(g_l \pm \frac{1}{2l+1}(g_s - g_l) \right) \mu_N$

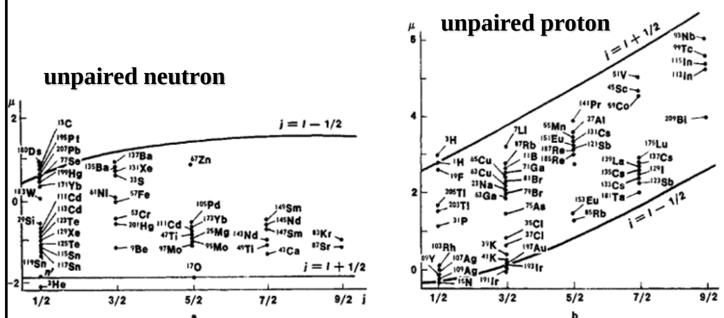
	proton	neutron
g_l	1	0
g_s	5,586	–3,826

Four cases:

unpaired proton	unpaired neutron
$j = l+1/2$	$j = l+1/2$
$j = l–1/2$	$j = l–1/2$

Nuclear magnetic moments (contd.)

„Schmidt lines"



The Schmidt-lines – originating from the shell model – are the limiting values. → The shell model is an extreme approximation.

The nuclear shell model (contd.)

Nuclear quadrupole moments $Q_p = \langle \psi_j | 3z^2 - r^2 | \psi_j \rangle = \langle \psi_j | r^2 Y_2^0(\vartheta,\varphi) | \psi_j \rangle$

The quadrupole moment of the unpaired proton:

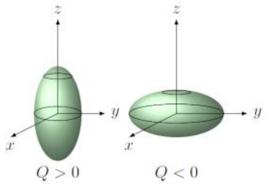
$$Q_p = \langle \psi_j | 3z^2 - r^2 | \psi_j \rangle = -\frac{2j-1}{2(j+1)} \langle \psi_j | r^2 | \psi_j \rangle = -\frac{2j-1}{2j+2} \cdot \frac{3}{5} R^2 = -\frac{2j-1}{2j+2} A^{\frac{2}{3}} \cdot 8{,}64\,[\mathrm{mb}]$$

For a „proton hole": $Q_{-p} = -Q_p$

Experimental fact: $|Q_{\exp}| >> |Q_p|$

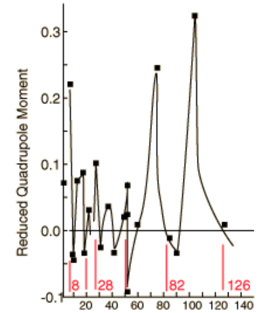
(sometimes even a factor of 30)!

Nuclide	Z	N	Character	Q^{exp} [mb]	Q^{SM} [mb]	Ratio
^{17}O	8	9	+1 neutron j = 5/2	−26	−33	0.8 ☺
^{39}K	19	20	− 1 proton j = 3/2	+55	+40	1.4 ☺
^{175}Lu	71	104	mid shell j = 7/2	+5600	−180	−31.1 ☹
^{209}Bi	83	126	+1 proton j = 9/2	−350	−221	1.6 ☺



The nuclear shell model (contd.)

Nuclear quadrupole moments



http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/imgnuc/quadrupole.gif

Quadrupole measurements for odd-A nuclei. The horizontal axis is either neutron number or proton number, whichever is odd

For a shell model, those nuclei with closed shells should be spherically symmetric and have no quadrupole moment

Between magic numbers the quadrupole moments can be really high → the shell model can not explain that.

Nuclear models #4: The nuclear collective model

1) Indications

We have seen that the shell model cannot explain some observations:

- **Nuclei between the closed shells have quadrupole moments even in the ground state.**
- **How can the ground state nuclear quadrupole moments be much larger than predicted by the shell model?**
- **Why are the magnetic moments differing from the Schmidt-lines?**
- **..etc.**

2) There were some simplification in the single-particle shell model:

The nuclear Hamiltonian: $\hat{H} = \sum_{i=1}^{A} \left(-\frac{\hbar^2}{2m_i} \Delta_i + \tilde{V}(\mathbf{r}_i) \right)$

single-particle operators **„mean" nuclear potential**

Assumption was: the shape of the „mean" nuclear potential is…

- **…central → depends only on the absolute value of r_i: $V(r_i)$ (true?)**
- **…similar shape as the nuclear density (since the nucleons create it). Is that necessarily spherical?**

Possible solution:

- **Keep the assumption that the shape is similar to the shape of the nuclear density (since the potential originates from the nucleons)**
- **Allow the nuclear density to be deformed ⟹ potential gets also deformed**
- **Keep the nuclear volume constant (incompressible fluid)**

How to determine the deformation?

„Hartree-Fock (Bogoliubov) method”

Iterative approach:

Nuclear density $\rho(\mathbf{r})$ → Nuclear potential $V(\mathbf{r})$ → Determine Ψ by minimizing the energy $\langle\Psi|H|\Psi\rangle$ → $|\Psi(\mathbf{r})|^2$ → Nuclear density $\rho(\mathbf{r})$

Note: „r” is a vector, it depends on the direction as well!

$\mathbf{r} = (r, \vartheta, \varphi)$

It converges only slowly (if converging at all) for many particles. Slow, difficult and cumbersome calculations (antisymmetrizing, pairing, etc.)

Rainwater approximation

„Marriage” of the liquid drop model and the shell model.

Concept: nucleus = core + valence nucleon(s)

core ↑ deformable liquid drop; valence nucleon(s) ↑ shell model

Leo James Rainwater (1917-1986) Nobel-prize 1975

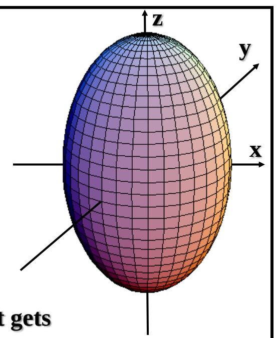
Core $\rho(\mathbf{r})$ → Nuclear potential $V(\mathbf{r})$ → Nucleon energy variation by perturbation → Nucleon deforms the core → Energy minimisation → Core $\rho(\mathbf{r})$

Rainwater approximation (contd.)

Simplest case:

- Only one nucleon outside the core
- The core can be deformed to ellipsoid
- Volume conserved: $\frac{4\pi}{3}R^3 = \frac{4\pi}{3}A^2 B$

For small deformations: $A = R\left(1 - \frac{1}{3}\varepsilon\right)$, $B = R\left(1 + \frac{2}{3}\varepsilon\right)$



The potential is originally spherical, but it gets also deformed: $V(r) \to V(r(\vartheta))$,

where $r(\vartheta) = r\left[1 + \frac{\varepsilon}{3}\left(3\cos^2\vartheta - 1\right)\right] \Rightarrow r' = r(\vartheta)\left[1 - \frac{\varepsilon}{3}\left(3\cos^2\vartheta - 1\right)\right]$

The perturbed potential: $\delta V(r) = V(r') - V(r) \cong -\frac{\varepsilon}{3}\cdot\frac{dV}{dr}\cdot r\left(3\cos^2\vartheta - 1\right)$

(to the first order)

Rainwater approximation (contd.)

$$\boxed{\delta V = -\frac{\varepsilon}{3}\cdot\frac{dV}{dr}\cdot r\left(3\cos^2\vartheta - 1\right)}$$

The total energy of the nucleus $E = \underbrace{E_C}_{\text{core}} + \underbrace{E_N}_{\text{valence nucleon}}$

a) The unperturbed core is spherical: $E_C = E_C^{(0)} + \frac{1}{2}S\cdot\varepsilon^2 + \ldots$

where $S = \left.\frac{\partial^2 E_C}{\partial\varepsilon^2}\right|_0$ is the „rigidity” of the core.

b) The perturbed energy of the valence nucleon:

$$E_N = E_N^{(0)} + \left(\frac{dE_N}{d\varepsilon}\right)_0 \cdot\varepsilon + \ldots = E_N^{(0)} + \delta E_N^{(1)}$$

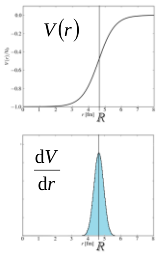
$\delta E_N^{(1)} = \int \psi^* \delta V \psi \cdot d^3\mathbf{r}$ (perturbation theory)

Here ψ is the unperturbed wave function: $\psi(r,\vartheta,\varphi) = \frac{\mathcal{R}(r)}{r} Y_l^m(\vartheta,\varphi)$

Therefore $\delta E_N^{(1)} = -\frac{\varepsilon}{3}\int |\mathcal{R}(r)|^2 \frac{\mathrm{d}V}{\mathrm{d}r} r\cdot\mathrm{d}r \int |Y|^2 \left(3\cos^2\vartheta - 1\right)\cdot\mathrm{d}\Omega$

Note: since $|\psi|^2$ is density, its dimension is $[\mathrm{m}^{-3}]$.

⇨ The dimension of $|\mathcal{R}(r)|^2$ is $[\mathrm{m}^{-1}]$.



$$\delta E_N^{(1)} = -\frac{\varepsilon}{3}\int |\mathcal{R}(r)|^2 \frac{\mathrm{d}V}{\mathrm{d}r} r\cdot \mathrm{d}r \int |Y|^2 \left(3\cos^2\vartheta - 1\right)\cdot \mathrm{d}\Omega$$

therefore $\frac{\mathrm{d}V}{\mathrm{d}r} \approx V_0 \cdot \delta(r-R) \longrightarrow \int |\mathcal{R}(r)|^2 \frac{\mathrm{d}V}{\mathrm{d}r} r\cdot \mathrm{d}r \approx V_0 |\mathcal{R}(R)|^2 R$

The „mass" quadrupole moment of the nucleon:

$$Q_N = \int \psi^* r^2 \left(3\cos^2\vartheta - 1\right)\psi \cdot \mathrm{d}^3\mathbf{r} = \int |\mathcal{R}(r)|^2 r^2 \mathrm{d}r \int |Y|^2 \left(3\cos^2\vartheta - 1\right)\cdot \mathrm{d}\Omega$$

$$\langle r^2 \rangle = \int |\mathcal{R}(r)|^2 r^2 \mathrm{d}r$$

We get $\delta E_N^{(1)} = -\frac{\varepsilon}{3} V_0 |\mathcal{R}(R)|^2 R \frac{Q_N}{\langle r^2\rangle} = -\varepsilon \cdot U_0 \frac{Q_N}{\langle r^2\rangle}$ **where** $U_0 = V_0 \frac{|\mathcal{R}(R)|^2 \cdot R}{3}$

The energy of the nucleus: $E = E_C + E_N = E_C^{(0)} + \frac{1}{2} S\cdot\varepsilon^2 + E_N^{(0)} - \varepsilon\cdot U_0 \frac{Q_N}{\langle r^2\rangle}$

$\Delta E = \frac{1}{2} S\cdot\varepsilon^2 - \varepsilon\cdot U_0 \frac{Q_N}{\langle r^2\rangle}$ **This has a minimum, if** $\boxed{\varepsilon = \frac{U_0}{S}\frac{Q_N}{\langle r^2\rangle}}$

Note that the sign of the deformation is the same as the sign of Q_N:
The nucleon „attracts" the core, and deforms it.

The total quadrupole moment of the nucleus: $Q = Q_C + Q_N$

If we know the ε, the electric quadrupole moment of the core can be calculated: $Q_C = \int \rho\cdot r^2\left(3\cos^2 - 1\right) d^3\mathbf{r}$ **where ρ is the proton-density.**

Using $Z = \frac{4\pi}{3} A^2 B \rho_0$ **we get** $Q_C = \frac{3Z}{4\pi A^2 B}\int \left(\frac{\rho(\mathbf{r})}{\rho_0}\right) r^2 \left(3\cos^2 - 1\right) d^3\mathbf{r} \approx \frac{4}{5} Z R^2 \varepsilon$

Finally we have: $\boxed{Q = \left(1 + \frac{4}{5}\frac{ZR^2}{\langle r^2\rangle}\cdot\frac{U_0}{S}\right) Q_N}$

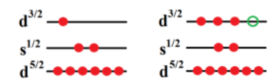
$Q >> Q_N$ is possible for small S („soft" core, easily deformable)

Few examples:

$^{209}_{83}\mathrm{Bi}_{126}$ **doubly magic core (rigid) + 1 proton:** $\frac{Q_{meas}}{Q_N} \approx 1$

$^{85}_{37}\mathrm{Rb}_{48} \longrightarrow Q = 0{,}31\,\mathrm{b}$ $^{87}_{37}\mathrm{Rb}_{50} \longrightarrow Q = 0{,}14\,\mathrm{b}$ **(magic number of neutrons)**

$^{33}_{16}\mathrm{S}_{17} \longrightarrow Q = -0{,}06\,\mathrm{b}$ $^{35}_{16}\mathrm{S}_{19} \longrightarrow Q = +0{,}04\,\mathrm{b}$

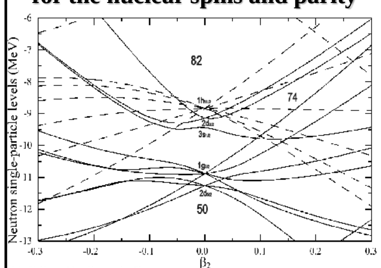


Consequences of core deformation (and potential deformation)

1) **Nilsson level schemes**

The degenerated levels split with deformation $\longrightarrow$ magic numbers may change!

2) This gives a better description for the nuclear spins and parity



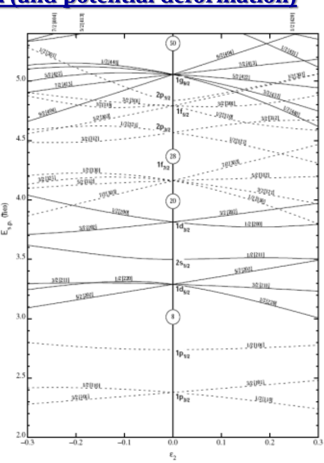


Figure 4. Nilsson diagram for protons or neutrons, Z or N ≤ 50 (ε_4 = 0).