Nuclear Physics
(8th lecture)

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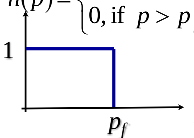
The Fermi-gas model

Since the nucleons are fermions, we can try to set up a model of fermion „gas”.

Assumption: the fermions are „closed” in a spherical potential well, but they move „freely” inside .

The number of states in the phase-space:

$$n(p)=\begin{cases}1, \text{if } p\le p_f\\ 0, \text{if } p>p_f\end{cases}$$



$$\mathrm{d}\mathcal{N}=2\cdot\frac{\mathrm{d}^3\mathbf{r}\cdot\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3}=2\cdot\mathrm{d}^3\mathbf{r}\cdot\frac{p^2\mathrm{d}p}{h^3}\mathrm{d}\Omega_p$$

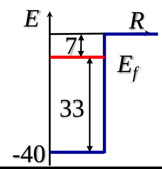
$$\frac{\mathcal{N}}{V}=\frac{2}{h^3}\cdot\frac{4\pi p_f^3}{3}\Rightarrow p_f^3=h^3\frac{3}{8\pi}\cdot\frac{\mathcal{N}}{V}$$

But: $V=\frac{4\pi}{3}R^3=\frac{4\pi}{3}r_0^3\cdot A$ **From this follows:** $p_{fp}=\left(\frac{9}{32\pi^2}\right)^{1/3}\cdot\frac{h}{r_0}\cdot\left(\frac{Z}{A}\right)^{1/3}$

Approximating: $\frac{Z}{A}\approx\frac{N}{A}\approx\frac{1}{2}$

The highest kinetic energy at the Fermi-level:

$$E_f=\frac{p_f^2}{2M}\approx 33\,\mathrm{MeV}$$



The depth of the nuclear potential is ~ 40 MeV for every nucleus!

$p_f=h\cdot\left(\frac{3\mathcal{N}}{8\pi V}\right)^{1/3}$

The total kinetic energy of the nucleons:

$$E_k=2\frac{1}{(2\pi\hbar)^3}\int\frac{p^2}{2M}n(p)\mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{p}=\frac{V}{h^3}\cdot\frac{2}{2M}\cdot\int_0^{p_f}p^2\cdot 4\pi p^2\mathrm{d}p=\frac{V}{h^3}\cdot\frac{4\pi}{M}\cdot\frac{p_f^5}{5}$$

The total kinetic energy of the protons and the neutrons:

$$\begin{cases}E_k^{(p)}=\frac{3}{10}\left(\frac{3\pi^2}{V}\right)^{2/3}\frac{\hbar^2}{M}Z^{5/3}\\ E_k^{(n)}=\frac{3}{10}\left(\frac{3\pi^2}{V}\right)^{2/3}\frac{\hbar^2}{M}N^{5/3}\end{cases}$$

The asymmetry energy:

We note that $\frac{N-Z}{A}<<\frac{N}{A}\approx\frac{Z}{A}\approx\frac{1}{2}$ **, and a Taylor-expansion yields:**

$$Z^{5/3}=A^{5/3}\left(\frac{N}{A}-\frac{N-Z}{A}\right)^{5/3}\approx A^{5/3}\left[\left(\frac{1}{2}\right)^{5/3}-\frac{5}{3}\left(\frac{N-Z}{A}\right)\cdot\left(\frac{1}{2}\right)^{2/3}+\frac{5}{9}\left(\frac{N-Z}{A}\right)^2\cdot\left(\frac{1}{2}\right)^{-1/3}-\ldots\right]$$

$$N^{5/3}=A^{5/3}\left(\frac{Z}{A}+\frac{N-Z}{A}\right)^{5/3}\approx A^{5/3}\left[\left(\frac{1}{2}\right)^{5/3}+\frac{5}{3}\left(\frac{N-Z}{A}\right)\cdot\left(\frac{1}{2}\right)^{2/3}+\frac{5}{9}\left(\frac{N-Z}{A}\right)^2\cdot\left(\frac{1}{2}\right)^{-1/3}-\ldots\right]$$

The total kinetic energy: $E_k=E_k^{(p)}+E_k^{(n)}=\frac{3}{10}\left(\frac{3\pi^2}{V}\right)^{2/3}\frac{\hbar^2}{M}\left(Z^{5/3}+N^{5/3}\right)$

$$E_k=\frac{3}{10}\left(3\pi^2\frac{A}{V}\right)^{2/3}\frac{\hbar^2}{M}A\cdot\left(\left(\frac{1}{2}\right)^{5/3}+\left(\frac{1}{2}\right)^{5/3}+\frac{5}{9}\left(\frac{N-Z}{A}\right)^2\left(\left(\frac{1}{2}\right)^{-1/3}+\left(\frac{1}{2}\right)^{-1/3}\right)\right)$$

$$\frac{A}{V}=\frac{3}{4\pi r_0^3}=\text{const.}\qquad 2\cdot\left(\frac{1}{2}\right)^{\frac{5}{3}}=0{,}63=\text{const.}\qquad 2\cdot\left(\frac{1}{2}\right)^{-\frac{1}{3}}=2{,}52=\text{const.}$$

The first part contributes to the volume term: $\propto A$

Note: it has positive sign, therefore it decreases the binding energy! (unlike in the Weizsäcker formula)

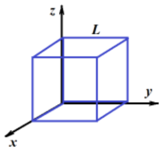
In the second part only the second order terms remain (the linear terms cancel): $\propto A\cdot\left(\frac{N-Z}{A}\right)^2=\frac{(N-Z)^2}{A}$

This way we get an asymmetry term, similar to the Weizsäcker formula's!

Note: the asymmetry term also has a positive sign! This means that it increases the kinetic energy, and therefore decreases the binding energy (like in the Weizsäcker formula).

The surface energy:

For simplicity consider now a cube (instead of a sphere)



The wave function: $\varphi(\mathbf{r}) = \sqrt{\frac{2^3}{V}} \sin(k_x x)\sin(k_y y)\sin(k_z z)$

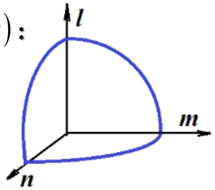
where $k_x = \frac{\pi}{L} n,\ \ k_y = \frac{\pi}{L} m,\ \ k_z = \frac{\pi}{L} \ell$ and $|\mathbf{k}^2| = k_x^2 + k_y^2 + k_z^2$

The number of states in the phase space $(V = L^3)$:

$$\mathcal{N} = A = 2\frac{V}{(2\pi)^3}\cdot\frac{4\pi K^3}{3} = 2\underbrace{\frac{1}{8}\cdot\frac{4\pi}{3}\left(\frac{LK}{\pi}\right)^3}$$

where $K = k_f$

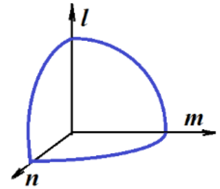
Volume of the 1/8 sphere in the space of (n, m, l) quantum numbers



So the total number of nucleons: $A \propto K^3$ from where: $K \propto A^{1/3}$

For $n = 0$, or $m = 0$, or $l = 0$ there is no wave function!
However, they were also included in the volume of 1/8 sphere!
A correction needed for the 3 quart-circle surface:

$$\mathcal{N} = A = 2\left[\frac{1}{8}\cdot\frac{4\pi}{3}\left(\frac{LK'}{\pi}\right)^3 - \frac{3}{4}\pi\left(\frac{LK'}{\pi}\right)^2\right]$$



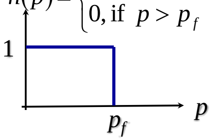
Obviously $K' > K$, i.e the kinetic energy will be increased!
Since the number of the states in the volume >> on the surface, at first approximation $K' \propto A^{1/3}$ → the correction term $\propto A^{2/3}$
The „**surface energy**" term can also be explained – at least qualitatively – in the Fermi-gas model.

Note: the attractive „**nuclear forces**", however, are not included and explained (they are introduced only as an „external" potential, holding together the Fermi-gas)!

The main assumption of the Fermi-gas model was, that the nucleons move „independently" – without interacting with each-other – in an outside potential well.
Is this not a crazy assumption for strongly interacting particles?

Answer:

$$n(p) = \begin{cases} 1, \text{if } p \le p_f \\ 0, \text{if } p > p_f \end{cases}$$



In the ground state of the nucleus every level is occupied up to the Fermi-level. Particles could scatter out only over the Fermi-level.

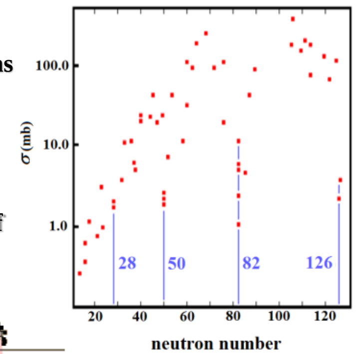
If a particle gets higher energy and momentum (outside the Fermi-level) in a scattering process, energy and momentum conservation would mean that the „other" particle should get lower energy/momentum → no empty state there, **forbidden**!

This shows also the **validity** of the Fermi-gas model: mainly the ground state properties

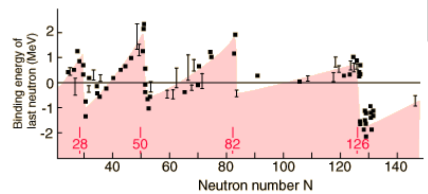
Nuclear shell model

1) Experimental indications

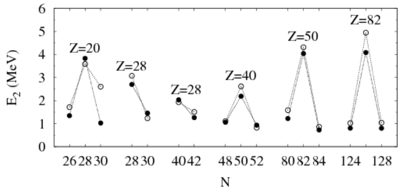
a) Neutron capture cross sections



b) Binding energy difference of the last neutron



c) Excitation energy of the first excited states:



Conclusion: nuclei with the following *Z* and/or *N* numbers are extremely stable (in comparison with their neighbours):

2, 8, 20, 28, 50, 82, 126 These are the „**magic numbers**".

Nuclei with magic *Z* or *N* numbers are called „**magic nuclei**"
If *Z* and *N* both are magic numbers, they are „**double magic nuclei**"

The following nuclei are double magic:

$^{4}_{2}\mathrm{He},\ ^{16}_{8}\mathrm{O},\ ^{40}_{20}\mathrm{Ca},\ ^{48}_{20}\mathrm{Ca},\ ^{48}_{28}\mathrm{Ni},\ ^{78}_{28}\mathrm{Ni},\ ^{208}_{82}\mathrm{Pb}$

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2) What characterizes a „shell"?

„Shell is a set of quantum-mechanical states with the same main quantum number" True?

Remember, in atomic physics:
The noble gases are (**experimental** fact): $_2\mathrm{He},\ _{10}\mathrm{Ne},\ _{18}\mathrm{Ar},\ _{36}\mathrm{Kr},\ _{54}\mathrm{Xe},\ _{86}\mathrm{Rn}$

„Magic numbers" in atomic physics should be: 2, 10, 18, 36, 54, 86.

The n^{th} „shell" can hold **theoretically** $2n^2$ electrons. Therefore the theoretical magic numbers would be:

n	$2n^2$	Magic number
1	2	2 (☺)
2	8	10 (☺)
3	18	28 (??)
4	32	60 (??)
5	50	110 (??)

There is a **discrepancy** between theory and experiment!
Conclusion: The magic numbers in Nature are **not** defined by the rule above!

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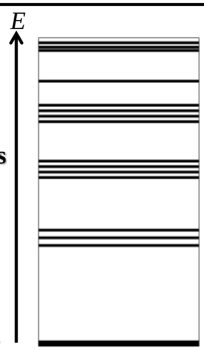
A quantum-mechanical system has a set of states with different energies, available for its constituents.

The excitation occurs if a particle goes into a **non-occupied** state at higher energy. If this state is „close" in energy, then the excitation is easy (requires small amount of energy). If the next empty state is „far" in energy, then the excitation is difficult.



Shell is a set of quantum-mechanical states with similar energy. Shells are separated by larger energy gaps.

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Shells in atomic physics (reminder):

H-atom (Bohr-model)

$$r = \left(\frac{\varepsilon_0 h^2}{m e^2 \pi}\right) \cdot n^2$$

This determines the „energy-shells"!

$$E = -\left(\frac{m e^4}{8 h^2 \varepsilon_0^2}\right) \cdot \frac{1}{n^2} \qquad \rightarrow n = 1,2,3\ldots$$

The quantum-mechanical treatment (Schrödinger-equation):

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi(r,\vartheta,\varphi) = E \cdot \psi(r,\vartheta,\varphi) \quad \text{(spherical coordinates)}$$

Wave function:

$$\psi_{n_r,l,m}(r,\vartheta,\varphi) = \frac{R_{n_r}(r)}{r} Y_l^m(\vartheta,\varphi) \rightarrow \begin{cases} n_r = 0,1,2,3\ldots \\ l = 0,1,2,3\ldots \\ -l \le m \le +l \end{cases}$$

What is the relation between *n* in the Bohr-model (defining the energy-shells), and the 3 quantum numbers of the wave-function?

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Shells in atomic physics (reminder):

Bohr — Quant. mech. — No. of particles

	n	n_r	l		$2(2l+1)$	
	1	0	0	1s	2	
2						2
	2	1	0	2s	2	
		0	1	2p	6	
10						10
	3	2	0	3s	2	
		1	1	3p	6	
18						
		0	2	3d	10	
						28
	4	3	0	4s	2	
		2	1	4p	6	
36						
		1	2	4d	10	
		0	3	4f	14	
						60

$n = n_r + l + 1$

This determines the „shells"!
But only in the H-atom!
(„True" Coulomb potential of a point-like nucleus)

Why are the shells different in the Periodic System?

Why are the shells different in the Periodic System?

Bohr:

$$r = \left(\frac{\varepsilon_0 h^2}{m e^2 \pi}\right) \cdot n^2$$

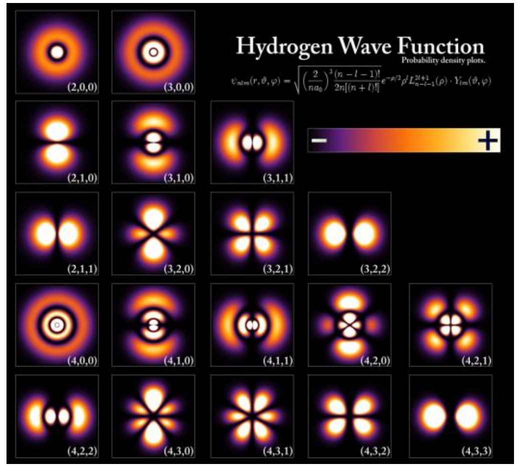
The „size" gets larger!

↓

The „screening" of the occupied inner states deforms the potential!

↓

Splitting of the states occur according to l !



The shell parameters depend on the shape of the potential!

3) The nuclear shell model

„The shell parameters depend on the shape of the potential!"
How is the „nuclear potential"?

The nuclear Hamiltonian:

$$\hat{H} = \sum_{i=1}^{A} -\frac{\hbar^2}{2m_i}\nabla_i^2 + \sum_{i \neq j} V_{i,j}(\mathbf{r}_i, \mathbf{r}_j)$$

single-particle operators — particle-particle interactions

A „trick":

$$\hat{H} = \sum_{i=1}^{A}\left(-\frac{\hbar^2}{2m_i}\nabla_i^2 + \tilde{V}(\mathbf{r}_i)\right) + \left[\sum_{i \neq j} V_{i,j}(\mathbf{r}_i, \mathbf{r}_j) - \sum_{i=1}^{A}\tilde{V}(\mathbf{r}_i)\right]$$

„mean" potential (for single-particles) — „residual" interaction (will be neglected)

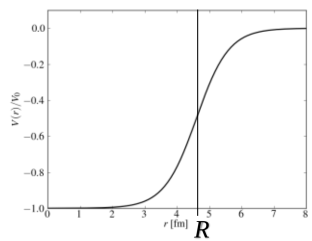
This is the „single-particle shell model"
Every particle moves independently in a mean potential

The nuclear shell model (contd.)

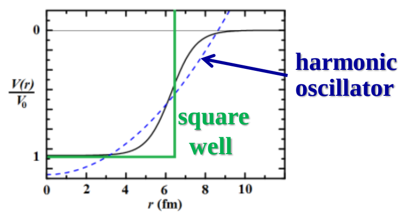
The shape of the „mean" nuclear potential.

- **Central → depends only on the absolute value of r_i: $V(r_i)$**
- **Similar shape as the nuclear density (since the nucleons create it)**

$$V(r) = -V_0\left(\frac{\rho(r)}{\rho_0}\right)$$



It is difficult to solve the Schrödinger-equation for this shape. Two different approximations:



The nuclear shell model (contd.)

Case #1: square well (infinite)! $V(r)=\begin{cases}-V_0, \text{ if } r\le R,\\ \infty, \text{ if } r>R\end{cases} \quad (V_0>0)$

Since the potential is central, the wave-function can be separated: $\psi(r,\Omega)=\frac{\phi(r)}{r}Y_l^m(\Omega)$

The Schrödinger equation in the interior region in spherical coordinate system is:

$$-\frac{\hbar^2}{2M}\Delta\psi(r,\Omega)-V_0\psi(r,\Omega)=E\psi(r,\Omega)$$

We get for the radial part:

$$\frac{d^2\phi_{n,\ell}}{dr^2}+\frac{2}{r}\cdot\frac{d\phi_{n,\ell}}{dr}+\left[\frac{2M}{\hbar^2}(V_0-E)-\frac{\ell(\ell+1)}{r^2}\right]\phi_{n,\ell}=0$$

Solution: Bessel functions: $\phi_{n,\ell}(r)=j_l(k_{n,\ell}r)$

where $k^2_{n,\ell}=\frac{2M}{\hbar^2}(V_0-E_{n,\ell})$

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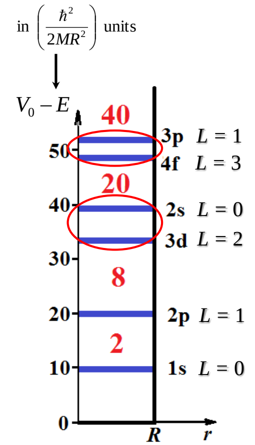
The nuclear shell model (contd.)

The functions have to go to 0 at $r=R$ **→ only roots of the Bessel functions are allowed →** $k_{n,l}$ **→ discrete values →** $E_{n,l}$ **discrete values too.**

kR		2(2L+1)	Sum	
π	L= 0 (1s)	2	2	2
4,49	L= 1 (2p)	6	8	8
5,79	L= 2 (3d)	10	18	
2π	L= 0 (2s)	2	20	20
6,99	L= 3 (4f)	14	34	?
7,22	L= 1 (3p)	6	40	

Second column: angular momentum quantum number (L) and the atomic physics notation of the state.



Note: inside a shell the higher L states go lower!

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The nuclear shell model (contd.)

Case #2 : the harmonic oscillator: $V(r)=-V_0+\frac{1}{2}m\omega^2r^2$

Easiest solution is in Descartes coordinates

Wave function (in one dimension): $\psi_n(x)=\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$

where $H_n(x)=(-1)^n e^{x^2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(e^{-x^2}\right)$ **are the Hermite-polinomials**

The energy spectrum: $E_n=\hbar\omega\left(n+\frac{1}{2}\right) \quad n=0,1,2,\ldots$

The 3D harmonic oscillator (in Descartes coordinates):

$$\psi_{n_x,n_y,n_z}(x,y,z)\propto e^{-\frac{m\omega(x^2+y^2+z^2)}{2\hbar}}H_{n_x}\left(\sqrt{\frac{m\omega}{\hbar}}x\right)H_{n_y}\left(\sqrt{\frac{m\omega}{\hbar}}y\right)H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right)$$

$E_n=\hbar\omega\left(n_x+n_y+n_z+\frac{3}{2}\right)=\hbar\omega\left(n+\frac{3}{2}\right)$ **where** $n=n_x+n_y+n_z$

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The nuclear shell model (contd.)

a) The parity of the states:

Since $\psi_{n_x}(x)\propto e^{-\frac{m\omega x^2}{2\hbar}}\cdot H_{n_x}\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$ **and** $H_{n_x}(-x)=(-1)^{n_x}H_{n_x}(x)$

We get: $\psi_{n_x}(-x)=(-1)^{n_x}\psi_{n_x}(x)$

In 3D: $\psi_{n_x,n_y,n_z}(-x,-y,-z)=(-1)^{n_x+n_y+n_z}\psi_{n_x,n_y,n_z}(x,y,z)$

Finally: $\psi_n(-\mathbf{r})=(-1)^n\psi_n(\mathbf{r})$

b) The number of degeneration of the states

$n=n_x+n_y+n_z$ **How many different combinations of n_x, n_y, n_z delivers the same n?**

$$g=\sum_{n_x=0}^{n}n-n_x+1=\frac{(n+1)(n+2)}{2}$$

The possible number of particles in a state: $2g=(n+1)(n+2)$ 20

The nuclear shell model (contd.)

The 3D harmonic oscillator (in spherical coordinates):

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi(r,\vartheta,\varphi) = E\cdot\psi(r,\vartheta,\varphi)$$

Wave function: $\psi_{n_r,l,m}(r,\vartheta,\varphi) = \frac{R_{n_r}(r)}{r} Y_l^m(\vartheta,\varphi) \rightarrow \begin{cases} n_r = 0,1,2,3... \\ l = 0,1,2,3... \\ -l \le m \le +l \end{cases}$

Question:

What is the relation between n in the Descartes-states (defining the energy-shells), and the 3 quantum numbers of the wave-function?

The idea for finding the correspondence: functions should be chosen to follow

- **the same parity behaviour and**
- **the same number of degenerations!**

The nuclear shell model (contd.)

Magic numbers: 2, 8, 20, 28, 50, 82, 126

Remember:

$\psi_n(-\mathbf{r}) = (-1)^n \psi_n(\mathbf{r})$

$2g = (n+1)(n+2)$

Descartes			spherical				
n	π	$(n+1)(n+2)$	n_r	l		$2(2l+1)$	
0	+	2	0	0	1s	2	2 ☺
1	–	6	0	1	2p	6	8 ☺
2	+	12	1	0	2s	2	
			0	2	3d	10	20☺
3	–	20	1	1	3p	6	
			0	3	4f	14	40☹
4	+	30	2	0	3s	2	
			1	2	4d	10	
			0	4	5g	18	70☹

$n = 2n_r + l$

The harmonic oscillator is a good „first” approximation, but something more should be included!