Nuclear Physics
(4th lecture)

Content

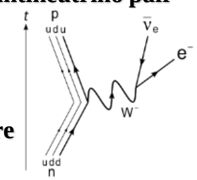
- Basic theory of beta decays
- Energy considerations
- Fermi theory of beta-decay
- Fermi and Gamow-Teller transitions

Basic theory of beta decay

Three goals:
- The energy (mass) condition for the decay to occur
- The shape of the electron (positron) spectrum (transition probability in function of electron energy)
- The integral transition probability (decay with any energy)

Starting point:
According to the quantum mechanics, no electron can be present inside the nucleus ⟶ the electron-antineutrino pair is created in the moment of the interaction!

Remark:
In particle physics we describe the beta-decay as the picture shows, but here we are now in classical nuclear physics!



$M(W^{\pm})$ = 80,385±0,015 GeV/c^2 ⟶ very short range

Energy considerations:

Negative β–decay: ${}^{A}_{Z}X \to {}^{A}_{Z+1}Y + e^- + \tilde{\nu}$

$$M(X_{nucl})\cdot c^2 = M(Y_{nucl})\cdot c^2 + m(e^-)\cdot c^2 + m(\tilde{\nu})\cdot c^2 + Q$$

$$\underbrace{[M(X_{nucl}) + Z\cdot m(e^-)]\cdot c^2}_{M(X_{atom})\cdot c^2} = \underbrace{[M(Y_{nucl}) + Z\cdot m(e^-)]\cdot c^2 + m(e^-)\cdot c^2}_{M(Y_{atom})\cdot c^2} + 0 + Q$$

We get for the decay energy: $Q = [M(X_{atom}) - M(Y_{atom})]\cdot c^2$

Positive β–decay: ${}^{A}_{Z}X \to {}^{A}_{Z-1}Y + e^+ + \nu$

$$M(X_{nucl})\cdot c^2 = M(Y_{nucl})\cdot c^2 + m(e^+)\cdot c^2 + m(\tilde{\nu})\cdot c^2 + Q$$

$$\underbrace{[M(X_{nucl}) + Z\cdot m(e^-)]\cdot c^2}_{M(X_{atom})\cdot c^2} = \underbrace{[M(Y_{nucl}) + Z\cdot m(e^-)]\cdot c^2}_{M(Y_{atom})\cdot c^2 + m(e^-)\cdot c^2} + m(e^+)\cdot c^2 + 0 + Q$$

Here the decay energy: $Q = [M(X_{atom}) - M(Y_{atom}) - 2\cdot m(e^{\pm})]\cdot c^2$

Positive β–decay occurs only, if

$$[M(X_{atom}) - M(Y_{atom})]\cdot c^2 > 2\cdot m(e^{\pm})\cdot c^2 = 1022\ \text{keV}$$

Electron capture (EC): ${}^{A}_{Z}X + e^- \to {}^{A}_{Z-1}Y + \nu$

$$M(X_{nucl})\cdot c^2 + m(e^-)\cdot c^2 = M(Y_{nucl})\cdot c^2 + m(\nu)\cdot c^2 + Q$$

$$\underbrace{[M(X_{nucl}) + Z\cdot m(e^-)]\cdot c^2}_{M(X_{atom})\cdot c^2} + m(e^-)\cdot c^2 = \underbrace{[M(Y_{nucl}) + (Z)\cdot m(e^-)]\cdot c^2}_{M(Y_{atom})\cdot c^2 + m(e^-)\cdot c^2} + 0 + Q$$

The electron masses cancel out, so we get again for the decay energy: $Q = [M(X_{atom}) - M(Y_{atom})]\cdot c^2$

Condition for energy (atomic masses) summarized:

Negative β–decay: ${}^{A}_{Z}X \to {}^{A}_{Z+1}Y + e^- + \tilde{\nu}$ ⟶ $M(X) - M(Y) > 0$

Positive β–decay: ${}^{A}_{Z}X \to {}^{A}_{Z-1}Y + e^+ + \nu$ ⟶ $M(X) - M(Y) > 2m(e)$

Electron capture: ${}^{A}_{Z}X + e^- \to {}^{A}_{Z+1}Y + \nu$ ⟶ $M(X) - M(Y) > 0$

The shape of the energy spectrum

Starting point:
Fermi's „Golden rule": $\lambda(E_f) = \frac{2\pi}{\hbar}|V_{i,f}|^2 \rho(E_f)$

Partial decay constant

Interaction matrix-element

Final state density: $\rho(E_f) = \frac{dn}{dE_f}$

Both factors can depend on the energy of the emitted particles
⟶ shape of the spectrum

Calculation of the matrix element

$$V_{i,f} = g\int \Psi_f^* \hat{\mathbf{H}}_\beta \Psi_i d^3\mathbf{r}$$

(H_β is the interaction operator, g is the interaction „strength")

The initial state is only a nucleus: $\Psi_i = \psi_i$

The final state is a nucleus + electron +(anti)neutrino: $\Psi_f^* = \psi_f^* \varphi_\nu^* \varphi_e^*$

The matrix element becomes: $V_{i,f} = g\int \psi_f^* \varphi_\nu^* \varphi_e^* \hat{\mathbf{H}}_\beta \psi_i d^3\mathbf{r}$

First approximation: $\varphi_e = \frac{1}{\sqrt{V}} e^{i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}}$ **and** $\varphi_\nu = \frac{1}{\sqrt{V}} e^{i\frac{\mathbf{q}\cdot\mathbf{r}}{\hbar}}$ **(plane waves)**
p is the momentum of the electron and q is that of the neutrino.
V is a normalization volume.

Further approximations:

- **the integral will extend only in the region of the nucleus, since the ψ nuclear wave functions are zero outside. But for this region $\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar} << 1$, so we get** $\varphi_e \approx \frac{1}{\sqrt{V}}\left(1 + i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar} + \ldots\right)$ **and** $\varphi_\nu \approx \frac{1}{\sqrt{V}}\left(1 + i\frac{\mathbf{q}\cdot\mathbf{r}}{\hbar} + \ldots\right)$
- **Further, we take only the first term:** $\varphi_e = \varphi_\nu \approx \frac{1}{\sqrt{V}}$

In this approximation the matrix element depends only on the nuclear states:

$$V_{i,f} = g\frac{1}{V}\int \psi_f^* \hat{\mathbf{H}}_\beta \psi_i d^3 r = g\frac{1}{V} M_{f,i}$$

In this approximation Fermi's Golden rule becomes:

$$\lambda(E_f) = \frac{2\pi}{\hbar} g^2 |M_{f,i}|^2 \frac{1}{V^2} \cdot \frac{dn}{dE_f}$$

↑ depends only on the nucleus

Calculation of the final density of states: $\rho(E_f) = \frac{dn}{dE_f}$

***The simplest assumptions*:**

1) **Since the decay is kinematically not complete (3 particles in the final state), the electron and the (anti)neutrino randomly „share" the energy available to them.**
2) **Since the nucleus is much heavier than the electron and the neutrino, the kinetic energy of the recoil can be neglected: all energy goes into the kinetic energy of the electron+neutrino**

1) What does randomly share mean?

Every microstate in the common phase-space will be populated with equal probability.

The number of electron states in the phase space (since the electrons are emitted in a random direction):

$$\mathrm{d}n_e = \frac{1}{(2\pi\hbar)^3} V \cdot 4\pi \cdot p^2 \cdot \mathrm{d}p$$

Similarly, the number of antineutrino states in the phase space:

$$\mathrm{d}n_\nu = \frac{1}{(2\pi\hbar)^3} V \cdot 4\pi \cdot q^2 \cdot \mathrm{d}q$$

The energy in the final state is: $E_f = E_e + E_\nu$

Where $E_e^2 = p^2c^2 + (mc^2)^2$ **and** $E_\nu^2 = (E_f - E_e)^2 = q^2c^2$

So we get $\frac{dn}{dE_f} = \frac{1}{(2\pi\hbar)^6}(4\pi V)^2 \cdot \frac{1}{c^2}\left(E_e^2 - (mc^2)^2\right) \cdot \frac{1}{c^2}(E_f - E_e)^2 \cdot dp \cdot \frac{dq}{dE_f}$

Using $\frac{dq}{dE_f} = \frac{1}{c}$ **(at a given E_e)** $\frac{dn}{dE_f} = V^2 \frac{(4\pi)^2}{(2\pi\hbar)^6 c^5} \cdot \left(E_e^2 - (mc^2)^2\right) \cdot (E_0 - E_e)^2 \cdot dp$

($E_f = E_0$ because electron and neutrino get the total decay energy)

We measure E_e and not p, so instead of dp we need dE_e

Using $pc = \sqrt{E_e^2 - (mc^2)^2}$ **we get** $dp = \frac{1}{c}\frac{E_e}{\sqrt{E_e^2 - (mc^2)^2}} \cdot dE_e$

Substituting it back: $\frac{dn}{dE_f} = V^2\left(\frac{4\pi}{(2\pi\hbar)^3 c^3}\right)^2 \cdot \sqrt{E_e^2 - (mc^2)^2} \cdot (E_0 - E_e)^2 \cdot E_e \cdot dE_e$

And finally: $\lambda(E_e) = g^2 \frac{1}{2\pi^3\hbar^7 c^6}|M_{f,i}|^2 \cdot \sqrt{E_e^2 - (mc^2)^2} \cdot (E_0 - E_e)^2 \cdot E_e \cdot dE_e$

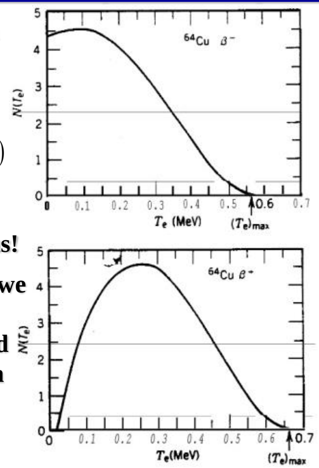
Comparison of the theoretical and experimental spectra

Note that in the above calculation there was no difference between negative and positive β–decay. However, experiment shows, that there is a difference! $\left({}^{64}_{29}\mathrm{Cu}\right)$

The difference is due to the Coulomb-interaction between the electron (positron) and the nucleus!

Solution: instead of plane-waves, we should use the Coulomb wave-functions for the outgoing charged particle. This leads to a correction factor: $F(Z,E_e)\propto \frac{\xi}{1-e^{-\xi}}$ **, where**

$\xi=\pm 2\pi\frac{Ze^2}{\hbar v}$ **,** v **is the velocity.**



R. D. Evans, The Atomic Nucleus (New York: McGraw-Hill, 1955)

9

The Fermi-Kurie plot

Finally we get $\lambda(E_e)=g^2\left|M_{f,i}\right|^2\cdot\frac{1}{2\pi^3\hbar^7c^6}\cdot F(Z,E_e)\cdot\sqrt{E_e^2-\left(mc^2\right)^2}\cdot\left(E_0-E_e\right)^2\cdot E_e\cdot dE_e$

Experimental test:

The detected counts in a given electron energy interval (E_e, E_e+dE_e):

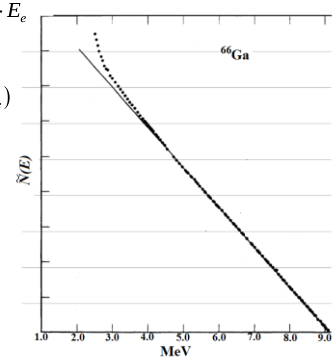
$$N(E_e)\propto F(Z,E_e)\cdot\sqrt{E_e^2-\left(mc^2\right)^2}\cdot\left(E_0-E_e\right)^2\cdot E_e$$

From this we get

$$\tilde{N}(E_e)=\left(\frac{N(E_e)}{F(Z,E_e)\cdot E_e\cdot\sqrt{E_e^2-\left(mc^2\right)^2}}\right)^{\frac{1}{2}}\propto\left(E_0-E_e\right)$$

The Fermi-Kurie plot is linear, if the previous assumptions are valid („allowed decays").

For „forbidden decays" the Taylor expansion $\varphi_e\approx\frac{1}{\sqrt{V}}\left(1+i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}+\ldots\right)$ **should continue ⟶ additional dependence on** p **(and energy).**



D. C. Camp and L. M. Langer, Phys.Rev. 129, 1782 (1963) 10

The total decay constant of the β–decay, the log(*ft*) value

The total decay constant is the integral of the partial: $\lambda=\int_0^{E_0}\lambda(E_e)dE_e$

$$\lambda=g^2\left|M_{f,j}\right|^2\cdot\frac{1}{2\pi^3\hbar^7c^6}\cdot\int_0^{E_0}F(Z,E_e)\cdot\sqrt{E_e^2-\left(mc^2\right)^2}\cdot\left(E_0-E_e\right)^2\cdot E_e\cdot dE_e$$

Define the „Fermi integral" (its values are tabulated):

$$f(Z,E_0)=\frac{1}{\left(mc^2\right)^5}\int_0^{E_0}F(Z,E_e)\cdot\sqrt{E_e^2-\left(mc^2\right)^2}\cdot\left(E_0-E_e\right)^2\cdot E_e\cdot dE_e$$

$$\lambda=g^2\left|M_{f,j}\right|^2\cdot\frac{1}{2\pi^3\hbar^7c^6}\cdot f(Z,E_0)\cdot\left(mc^2\right)^5$$

Taking into account that $\lambda=\frac{\ln 2}{t}$ **the** $f(Z,E_0)\cdot t$ **(**$f\cdot t$ **further on)**

$$f\cdot t=\ln 2\frac{2\pi^3\hbar^7}{g^2m^5c^4\left|M_{f,i}\right|^2}$$

This is depending only on the $M_{f,i}$

Since t **can range through many orders of magnitude, usually one uses the** $\log_{10}(f\cdot t)$ **value (**t **should be given in seconds)**

11

The strength of the weak interaction

For β–decays with shortest lifetimes $\log_{10}(f\cdot t)\approx 3\ldots4$

These are the „super-allowed" decays. Among them those are the most interesting ones, where $0^+\rightarrow 0^+$ super-allowed decay occurs. For these $M_{f,i}=\sqrt{2}$ **(theoretically), independent of the nucleus! The experiment confirms it! The measured value:** $f\cdot t=3090\pm 5\,[\mathrm{s}]$

Using now $f\cdot t=\ln 2\frac{2\pi^3\hbar^7}{g^2m^5c^4\left|M_{f,i}\right|^2}$**, the value of** g **can be determined:**

$$g=8{,}8\cdot 10^{-5}\,\left[\mathrm{MeV}\cdot\mathrm{fm}^3\right]$$

Comparing the strength of the four interactions

With fundamental constants we create dimensionless quantities. For example for the weak interaction: $G=g\frac{M_n^2c}{\hbar^3}\approx 10^{-5}$

Similar dimensionless quantities:

- **Strong interaction:** **~1**
- **Electromagnetic interaction: 1/137** **~10^{-2}**
- **Weak interaction:** **~10^{-5}**
- **Gravity:** **~10^{-39}**

12

Remaining problems

- H_β = ???
- neutrino ⟺ antineutrino??
- electron ⟺ positron??

} Relativistic quantum mechanics???

Solution

In relativistic quantum mechanics the electron wave-function is a spinor with 4 components (describing the electron and the positron and the two spin positions)

$$\varphi_e = \begin{pmatrix} e^-_\uparrow \\ e^-_\downarrow \\ e^+_\uparrow \\ e^+_\downarrow \end{pmatrix}$$

Similarly, the neutrino wave-function is also a spinor with 4 components (describing the neutrino and the antineutrino and the two spin positions)

$$\varphi_\nu = \begin{pmatrix} \nu_\uparrow \\ \nu_\downarrow \\ \tilde{\nu}_\uparrow \\ \tilde{\nu}_\downarrow \end{pmatrix}$$

Remember: $V_{i,f} = g\int \psi_f^* \varphi_\nu^* \varphi_e^* \hat{\mathbf{H}}_\beta \psi_i d^3\mathbf{r}$

Therefore the product $\varphi_\nu^* \varphi_e^*$ in the integral may have 16 components (any combination = 4^2)!

The H_β operator can act differently on these components

⇩

The H_β operator also has 16 components!

Using the superposition principle

16 components ⟹ another 16 components!

(…of quantities with known symmetry)

Behaviour	Number of components
Scalar	1
Pseudoscalar	1
Vector	4
Axial-vector	4
Antisymmetric tensor	6

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{pmatrix}$$

Therefore the H_β operator also consists of 5 terms:

$H_\beta = H_{\beta,S} + H_{\beta,P} + H_{\beta,V} + H_{\beta,A} + H_{\beta,T}$

scalar, pseudoscalar, vector, axial vector, tensor (antisymmetric)

- Pseudoscalar can NOT make any transition! (can be shown)

Angular momentum and selection rules

$${}^{A}_{Z}X \rightarrow {}^{A}_{Z+1}Y + e^- + \tilde{\nu}$$

$\mathbf{J}_X = \mathbf{J}_Y + \mathbf{s}_e + \mathbf{s}_\nu$ vector equation. Denote $\mathbf{s}_e + \mathbf{s}_\nu = \mathbf{j}$

s_e =1/2, and s_ν =1/2, so $j = \begin{cases} 1, & (j_z = -1,\ 0,\ 1) \quad \uparrow\uparrow \\ 0, & (j_z = 0) \quad \uparrow\downarrow \end{cases}$

Can be shown:

- scalar and vector causes: j = 0 (Fermi-transitions)
- axial vector and tensor causes: j = 1 (Gamow-Teller transitions)

Selection rules: $J_Y = J_X$ for Fermi-transitions ($\Delta J = 0$)
$J_Y = J_X \pm 1$, or $J_Y = J_X$ for Gamow-Teller-transitions ($\Delta J = 0, \pm 1$)