Nuclear Physics
(3rd lecture)

Content

- The mass and the binding energy of the nucleus
- Liquid drop model, Weizsäcker formula
- Radioactive decay types and their characteristics
- Exponential decay law, half life, activity
- Radioactive decay chains, radioactive equilibrium
- Radioactive dating

The mass of the nucleus

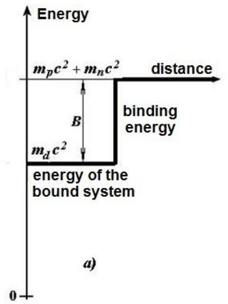
$M(A,Z) = Z \cdot m_{proton} + (A-Z) \cdot m_{neutron} - \Delta M$ (measurements show)
ΔM is called: mass defect

The protons and the neutrons are bound in the nucleus, and they can be freed only by investing some B binding energy.
Einstein: $E = mc^2$, which is for this case: $B = \Delta M \cdot c^2$
By measuring the mass (mass defect) precisely, the binding energy of the nucleus can be determined!

Measuring the mass:
With mass spectrometers (mass spectroscopes)
We saw a few examples
- for the devices,
- for the method (mass-doublet)

Energy and binding energy

Einstein: $E = m \cdot c^2$.
Since $m \geq 0$, therefore the total energy is $E \geq 0$.
For example: mass and energy of the deuteron (^2H)
$m_d = m_p + m_n - \Delta M$ (multiply by c^2)
$m_d c^2 = m_p c^2 + m_n c^2 - B$ (figure)



Energy and binding energy

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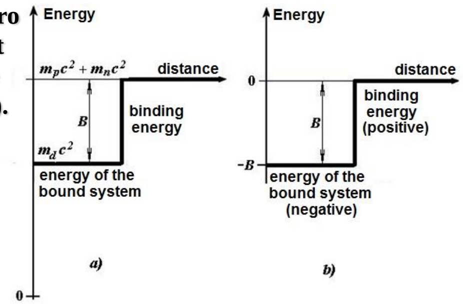
$m_d c^2 = m_p c^2 + m_n c^2 - B$ (left hand side of the figure)

However, usually the zero level of the energy is put at the energy of the free system (right hand side).

Then the energy of the bound system becomes NEGATIVE.
$E = -B$
(The binding energy is still positive!)

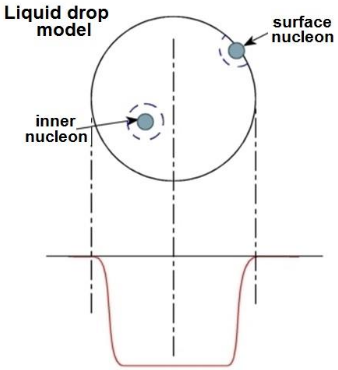


Binding energy of the nucleus (liquid drop model)

(Semi-empirical formula of Weizsäcker)

Model: since the nuclear density ~ constant, therefore the nucleus is similar to an (electrically charge) liquid drop

The nucleons interact only with their „neighbours”. If every nucleon was „inner”, then $B = b_V \cdot A$ would be. (b_V is the binding energy of an „inner” nucleon.)

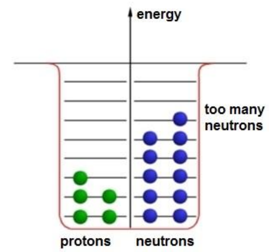


The nucleons on the surface are less bound, therefore
$B = b_V \cdot A - \beta \cdot 4\pi R^2$
Here β is a constant.

So far we considered only the (attractive) nuclear interaction. The nucleus has *Ze* charge, therefore the Coulomb-energy makes the binding weaker, because of the repulsion of the protons:

$$B = b_V A - \beta \cdot 4\pi R^2 - \frac{3}{5} k \frac{Z^2 e^2}{R}$$

Consider now that the protons and the neutrons are fermions, so the Pauli-principle is valid (2 identical particles can be in a state, at most). Therefore, too many neutrons or protons (asymmetry) makes the binding weaker (asymmetry energy):



$$B = b_V A - \beta \cdot 4\pi R^2 - \frac{3}{5} k \frac{Z^2 e^2}{R} - b_A \frac{(N-Z)^2}{A}$$

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Finally: empirical fact is that nuclei are stronger bound, if the number of protons and/or the number of neutrons are even (pairing energy)

$$B = b_V A - \beta \cdot 4\pi R^2 - \frac{3}{5} k \frac{Z^2 e^2}{R} - b_A \frac{(N-Z)^2}{A} + b_P \delta \cdot A^{-1/2}$$

Here δ = 1, if the nucleus is even-even
δ = 0, if the nucleus is even-odd
δ = –1, if the nucleus is odd-odd

Use now that $R = r_0 \cdot A^{1/3}$, and concatenate several constants into new constants:

$$\boxed{B = b_V A - b_F \cdot A^{2/3} - b_C \cdot \frac{Z^2}{A^{1/3}} - b_A \cdot \frac{(N-Z)^2}{A} + b_P \cdot \delta \cdot A^{-1/2}}$$

This is the semi-empirical binding energy formula of Weizsäcker

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The name of the 5 terms and the values of the constants

Parameter	Energy (J)	Energy (MeV)
Volume energy (b_V)	$2{,}52 \cdot 10^{-12}$	15,75
Surface energy (b_F)	$2{,}85 \cdot 10^{-12}$	17,8
Coulomb-energy (b_C)	$0{,}11 \cdot 10^{-12}$	0,711
Asymmetry energy (b_A)	$3{,}79 \cdot 10^{-12}$	23,7
Pairing energy (b_P)	$1{,}92 \cdot 10^{-12}$	12

These constants were fitted empirically.
With these 5 constants the binding energies of the more than 2000 known nuclei can be described with about 1-2% precision!!!

Introducing new, useful terms:
The „average” binding energy of a nucleon: $b = B/A$.

How „deeply” sits a nucleon in average in the nucleus?
How much is the average energy of a nucleon?
$\varepsilon = -b = -B/A$.

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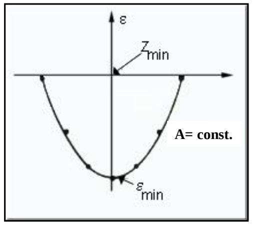
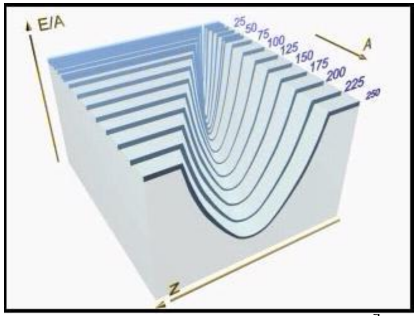
ε = –b = – B/A. Since B is a function of the (Z,A) composition, therefore ε is too: ε = ε(Z,A).
This can be visualized as a surface.

$$\varepsilon(Z,A) = -b_V + b_F \cdot A^{-1/3} + b_C \cdot \frac{Z^2}{A^{4/3}} + b_A \cdot \frac{(N-Z)^2}{A^2} - b_P \cdot \delta \cdot A^{-3/2}$$

Note, that the A = const. „cuts" are parables!





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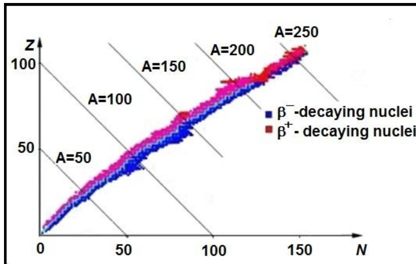
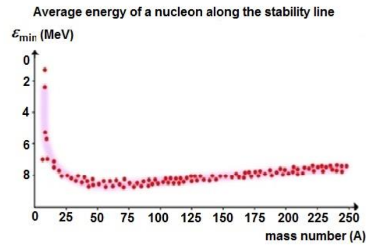


The space of Z_{min} on the (N,Z) „map"

This helps to understand the radioactive decays!

ε_{min} in function of the (A) mass number

This helps to understand the ways of nuclear energy production



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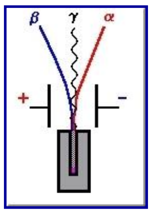
Radioactive decays

- **α – particles: ^4_2He nuclei**
- **β – particles: high energy electrons**
- **γ - radiation: electromagnetic (photons)**



Nuclei transform into each-other

conserved quantities

- **energy ($E = mc^2$ taking into account)**
- **nucleonic number (A)**
- **electric charge (nucleus: $+Ze$, electron: $-e$)**

Energy conservation

$a \longrightarrow b + c + \ldots$ $\quad \boxed{M_a \cdot c^2 = M_b \cdot c^2 + M_c \cdot c^2 + \ldots + Q}$

Energy condition of the decay: $Q > 0$, i.e. $M_a > M_b + M_c + \ldots$

Q is the energy „liberated" in the decay (it appears in the form of kinetic energy of the products)

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Conserved quantities (contd.)

- **energy ($E = mc^2$ taking into account)**
- **nucleonic number (A)**
- **electric charge**

1) α decay: $\;{}^A_Z\text{X} \rightarrow {}^4_2\text{He} + {}^{A-4}_{Z-2}\text{Y}$

Nucleonic number conserved ($A = 4+A-4$)

Term for Y : daughter nucleus

charge is conserved ($Z = 2 + Z-2$)

2) β decays:

a) Negative, β⁻ decay: $\;{}^A_Z X \rightarrow {}^A_{Z+1}Y + e^- + \tilde{\upsilon}$

$\left(e^- = {}^0_{-1}e\right)$

electron antineutrino

b) Positive, β⁺ decay: $\;{}^A_Z X \rightarrow {}^A_{Z-1}Y + e^+ + \upsilon$

positron neutrino

c) Electron capture (EC) : $\;{}^A_Z X + e^- \rightarrow {}^A_{Z-1}Y + \upsilon$

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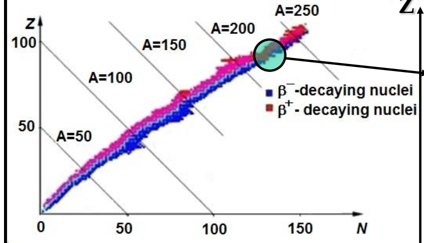
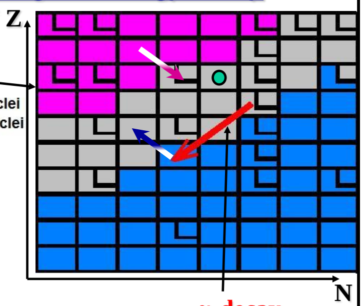
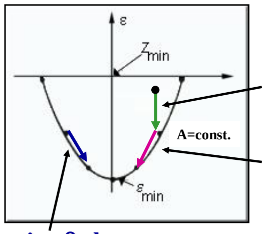
Radioactivity: walk along the energy valley







γ–decay

α-decay

$Z \rightarrow Z-2$

$N \rightarrow N-2$

positive β-decay,
or electron capture

negative β-decay

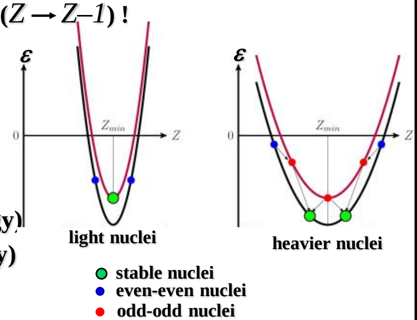
Interesting fact:

The ^{40}K, ^{92}Nb etc. nuclei can decay by negative β-decay ($Z \rightarrow Z+1$), but also by electron capture ($Z \rightarrow Z-1$) !

How is that possible?

Cause: the pairing energy

A = const. **cut is a parable,**
but even A can be formed in two ways: ($A = Z + N$)
even Z + even N (lower energy)
odd Z + odd N (higher energy)



This is also the cause, that only 4 stable odd-odd nuclei exist:
^{2}H, ^{6}Li, ^{10}B, ^{14}N.
(According to the first 4 odd numbers: 1, 3, 5, 7)

Exponential decay law, half life, activity (1)

The number N of the radioactive nuclei in a radioactive material is decreasing, since they decay: $N(t)$ is decreasing function of time.

Let us denote by $(\lambda \cdot \Delta t)$ the probability that one single nucleus decays during Δt time !
Then from N nuclei $N \cdot \lambda \cdot \Delta t$ will decay during Δt time.
The change (decrease) of the number of radioactive nuclei:
$\Delta N = -N \cdot \lambda \cdot \Delta t.$

From this we get: $\frac{\Delta N}{\Delta t} = -\lambda \cdot N(t)$ **in $\Delta t \longrightarrow 0$ limit:**

$\frac{dN}{dt} = -\lambda \cdot N(t)$ **The solution is:** $\boxed{N(t) = N_0 \cdot e^{-\lambda t}}$

This is the exponential decay law

λ **is called: decay constant**
physical meaning of λ : decay probability in unit time

Exponential decay law, half life, activity (2)

Half life:

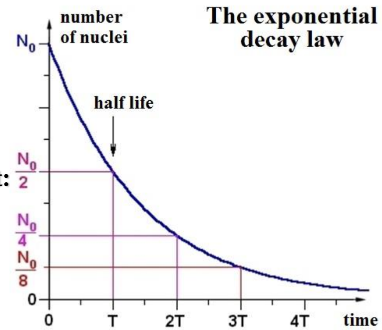
The T time, under which the number of nuclei decreases to half of its initial value:

$$N(T) = N_0 \cdot e^{-\lambda T} = \frac{N_0}{2}$$

From the second equation we get:

$$e^{\lambda T} = 2$$

Taking logarithm: $\boxed{T = \frac{\ln 2}{\lambda}}$



Activity:

Number of decays during unit time: $A = -\frac{dN}{dt}$

Using an above equation we get: $\boxed{A(t) = \lambda \cdot N(t)}$

The radioactive decay is a **statistical** process!
(we describe it by the decay **probability** λ in unit time)

- **For a given** atom it is not possible to forecast the exact time of its decay.
- The exponential decay law can be used only for **large number** of particles.

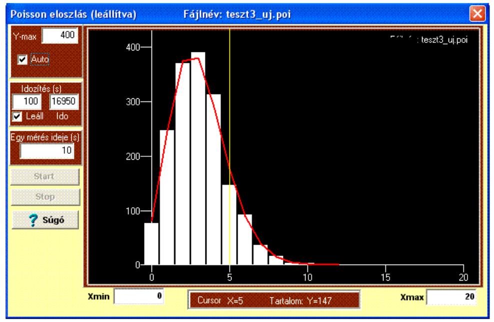
The **Poisson-distribution** gives the probability that during t time exactly k decays occur, if the activity of the source is a.
($t << T$, i.e. the decrease in the activity will be neglected; case of constant activity):

$$\boxed{P(k, at) = \frac{(at)^k}{k!} e^{-at}}$$

The Poisson-distribution

$$\boxed{P(k, at) = \frac{(at)^k}{k!} e^{-at}}$$

expectation value: $< k >= a \cdot t$
scatter: $\sigma_k = \sqrt{a \cdot t}$

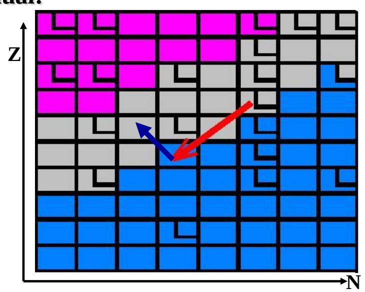


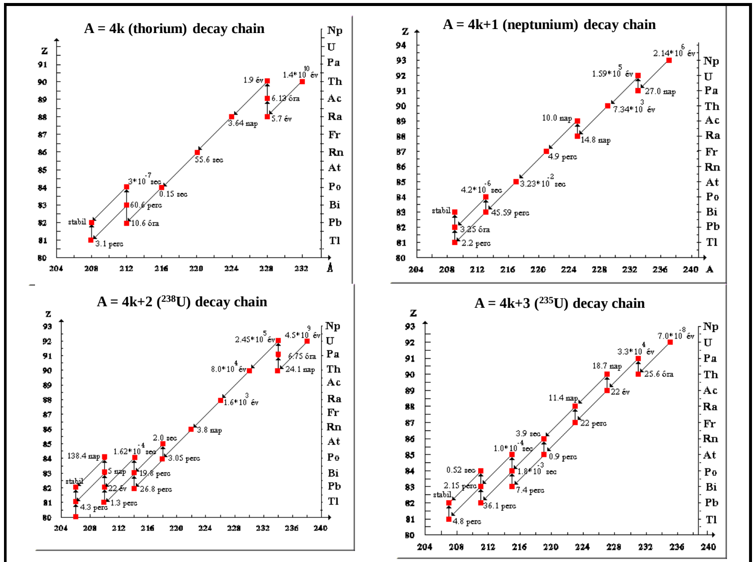
If $N = a \cdot t$ is the expectation value of the counts, then its scatter is:

$$\boxed{\sigma_N = \sqrt{N}}$$

Radioactive decay chains

If heavy radioactive nuclei decay, usually the daughter nuclei are also radioactive. These decay further, until finally the chain arrives to a stable nucleus.
Out of the possible radioactive decays, **only the alpha decay** changes the A mass number: decreases it by 4.
Consequence: dividing the mass number of every element in the chain by 4, we get the same residual!
Therefore there exist only 4 radioactive decay chains:
$A = 4k$, $A = 4k+1$,
$A = 4k+2$, $A = 4k+3$
The α-decays are followed by β-decays (and they are followed by γ-decays), so that the chain follows the bending of the energy valley.





Radioactive equilibrium

Consider a radioactive decay chain consisting only of 3 members:
1 ⟶ 2 ⟶ 3, with the decay constants: λ_1 and λ_2.
The number of the nuclei are respectively $N_1(t)$, $N_2(t)$, $N_3(t)$.
The equations describing the change in the number of nuclei:

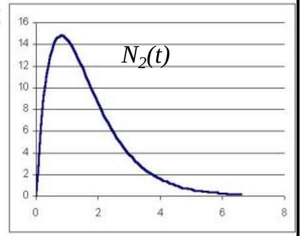
$$\frac{dN_1}{dt} = -\lambda_1 \cdot N_1(t) \qquad \text{(only decays)}$$

$$\frac{dN_2}{dt} = -\lambda_2 \cdot N_2(t) + \lambda_1 \cdot N_1(t) \qquad \text{(decays and created)}$$

$$\frac{dN_3}{dt} = +\lambda_2 \cdot N_2 \qquad \text{(only created from the previous)}$$

The solution of the first equation is known: $N_1(t) = N_{10} \cdot e^{-\lambda_1 \cdot t}$

The initial conditions for the solutions of the further equations:
$N_{20} = 0$ **and** $N_{30} = 0$**, i.e. there is nothing from the „2" and the „3" matter.**



The solution (see in the practice):

$$N_2(t) = N_1(0)\frac{\lambda_1}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

(if $\lambda_1 \neq \lambda_2$).

The activity of the „2" isotope :

$$a_2(t) = \lambda_2 \cdot N_2(t) = a_1(0)\frac{\lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

Rewriting it, using $a_1(t) = a_1(0) \cdot e^{-\lambda_1 t}$

$$a_2(t) = a_1(t)\frac{\lambda_2}{\lambda_2 - \lambda_1}\left(1 - e^{-(\lambda_2 - \lambda_1)t}\right)$$

$$\boxed{a_2(t) = a_1(t)\frac{\lambda_2}{\lambda_2 - \lambda_1}\left(1 - e^{-(\lambda_2 - \lambda_1)t}\right)}$$

Special cases:

1) If $\lambda_2 > \lambda_1$, then after sufficiently long time the exponential vanishes, $a_2(t) \approx a_1(t)\frac{\lambda_2}{\lambda_2 - \lambda_1}$ **, from where we get**

$$\frac{a_2(t)}{a_1(t)} = \frac{\lambda_2}{\lambda_2 - \lambda_1} = \text{const.}$$ **, i.e. not depending of time!**

This is called transitional equilibrium.

2) If $\lambda_2 >> \lambda_1$, then the condition for the transitional equilibrium is fulfilled, but λ_1 can be neglected in the denominator against λ_2

$$\frac{a_2(t)}{a_1(t)} = \frac{\lambda_2}{\lambda_2} = 1$$ **With other words:** $a_1(t) = a_2(t)$

It can similarly be shown, that in a decay chain consisting of many members after sufficiently long time $a_1(t) = a_2(t) = a_3(t) = \ldots$, if λ_1 is much smaller than the other decay constants.
We call this case secular equilibrium.

In secular equilibrium: $a_1(t) = a_2(t) = a_3(t) = \ldots$

Using $a(t) = \lambda \cdot N(t) = \frac{N(t)}{T}\ln 2$ **we get:**

$$\frac{N_1(t)}{T_1} = \frac{N_2(t)}{T_2} = \frac{N_3(t)}{T_3} = \ldots$$

Writing it in another way:

$$\boxed{N_1(t) : N_2(t) : N_3(t) \ \ldots = T_1 : T_2 : T_3 : \ldots}$$

In secular equilibrium the ratio of the quantities of the individual members equals the ratio of their half lives.

This enables the determination of long half lives!
(for example: the half life of ^{238}U is 4,5·10^9 (billion) years.)

A simple simulation of the radioactive decay chains can be performed at the following link:

Radioactive dating

Using the decay properties of a radioactive isotope we draw conclusion about the age of the sample.
An assumption for the „initial" condition should be made!
Most frequently used isotopes for radioactive dating:

Isotope	Half life	Abundance (to the stable)
^{3}H (tritium)	12,262 year	$1 \cdot 10^{-18}$
^{14}C (radiocarbon)	5568 year	$2 \cdot 10^{-12}$
^{40}K	$1{,}3 \cdot 10^{9}$ year	$1{,}19 \cdot 10^{-4}$
^{87}Rb	$5 \cdot 10^{10}$ year	0,278
^{238}U	$4{,}51 \cdot 10^{9}$ year	0,992739
^{235}U	$7{,}04 \cdot 10^{8}$ year	0,007204
^{232}Th	$1{,}39 \cdot 10^{10}$ year	1,0

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The determination of the age is the most accurate, if the half life of the used radioactive isotope is comparable to the age.

Geological dating (10 million years – few billion years)

- relative
- absolute

Relative dating (non-nuclear methods)
- paleontological (fossils in sedimentary rocks)
- based on the location in the geological section

Absolute dating (nuclear methods)
- Rubidium-strontium (Rb-Sr) method
- Lead-helium method (Th, or uranium chain)
- Potassium-argon method (K-Ar)

Rubidium-strontium method: $^{87}Rb \xrightarrow[\beta^-]{50 \text{ billion y}} {}^{87}Sr$

$\frac{^{87}Sr}{^{87}Rb}$ **ratio depends on the age of the rock**

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Lead-helium method: based on the radioactive decay chains
- From the ^{238}U we finally get ^{206}Pb with ? α-decays.
- From ^{232}Th we finally get ^{208}Pb with ? α-decays
- From ^{235}U we finally get ^{207}Pb with ? α-decays

Therefore helium accumulates in the rock.

***Difficulties*:**
- difficult to separate the lead isotopes
- usually all three decay chains are present in a rock
- an isotope of Rn (radon) is a member of every chain
it is a noble gas, easily escapes (diffuses away), the chain „breaks".

Potassium-argon method (T = 1,3 billion years)

$^{40}K \longrightarrow {}^{40}Ca$ (88%)

$^{40}K \longrightarrow {}^{40}Ar$ (12%)

The ratios $\frac{^{40}Ca}{^{40}K}$, and $\frac{^{40}Ar}{^{40}K}$ **are to be measured.**

***Difficulties*:**
- ^{40}Ca is very common, originates not only from decay of ^{40}K
- ^{40}Ar is noble gas, it can easily escape (diffuses away).

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Radiocarbon method (T = 5568 year)

The ^{14}C is continuously created in the atmosphere due to the cosmic radiation. Its equilibrium concentration (CO_2) in the air is $^{14}C/^{12}C = 1{,}2 \cdot 10^{-12}$. The plants take it up, and this way it gets also into the animals with their metabolism. When the plant or animal dies, the metabolism stops, and the ^{14}C will only decay.

Here t is the time after the death, T is the half life.

$$\frac{N(^{14}C)}{N(^{12}C)} = 1{,}2 \cdot 10^{-12} \left(\frac{1}{2}\right)^{\frac{t}{T}}$$

Tritium method (T = 12,26 year)

The ^{3}H is continuously created in the atmosphere due to the cosmic radiation. Its equilibrium concentration (H_2O) in the air is $^{3}H/^{1}H = 1 \cdot 10^{-18}$. This concentration will be maintained in the surface waters. The age of the underground water can be determined from the tritium concentration.

$$\frac{N(^{3}H)}{N(^{1}H)} = 1 \cdot 10^{-18} \left(\frac{1}{2}\right)^{\frac{t}{T}}$$

(The age of dead creatures can NOT be determined, since the H-exchange with the environment continues even after the death.)

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