Simulation of a radioactive decay chain

The radioactive nuclei are not „aging”, which means that it is not possible to determine how „old” is a selected atom (i.e. how long time ago was it created). Similarly, it is not possible to foresee exactly in which moment it will decay. Only probability statements can be made about their decay – similar to the lottery. The fact, that every moment is similar to them (they are not „aging”) means that the probability of their decay is the same for every small unit of time. In short, the decay probability per unit time is constant. This is called **decay constant** and denoted by λ. Since this is a probability „per unit time”, (probability density) therefore its unit is [1/s]. From the definition it follows that the probability of the decay of an atom over time Δt is: $\lambda \cdot \Delta t$.

Let us denote the number of radioactive atoms in the sample by $N(t)$ (N is a very large number). We expect that during Δt time $N(t) \cdot \lambda \cdot \Delta t$ atom will decay, therefore the number of the radioactive atoms will be reduced by this value. Mathematically: $\Delta N = -N(t) \cdot \lambda \cdot \Delta t$.

This equation can be rewritten as: $\frac{\Delta N}{N(t)} = -\lambda \cdot \Delta t$. (1)

The so called exponential decay law can easily be derived from this equation if we perform the $\Delta t \to 0$ limes, and then integrate both sides of the equation: $\int_{N_0}^{N(t)} \frac{dN'}{N'} = -\lambda \cdot \int_0^t dt'$. After performing the integration we get:

$\ln N(t) - \ln N_0 = -\lambda \cdot (t - 0)$.

This can be rewritten as: $\ln \frac{N(t)}{N_0} = -\lambda \cdot t$, from where we get:

$$N(t) = N_0 \cdot e^{-\lambda \cdot t}. \quad (2)$$

This is the **exponential decay law**.

> ***Note:*** *This gives only the* ***expected value*** *of the number of radioactive atoms in function of the time! In the actual cases there are some deviations from this, since – as we said – the decays are governed by statistical laws. It is also obvious from the (1) equation, that if $\Delta t \to 0$, then naturally also $\frac{\Delta N}{N(t)} \to 0$ should hold. However, since ΔN can only be integer (since „fractional atom” cannot exist), therefore the limes is meaningful only for very large $N(t)$! Radioactive materials used in everyday life generally met the criterion $\frac{\Delta N}{N(t)} \to 0$, because of the very large value of $N(t)$.*

In the program we simulate the decay of 2482 radioactive „nuclei". The initial nuclei are of **red** colour, the resulting daughter nuclei are **blue**. The simulation does **NOT use the exponential decay law** to calculate the number of nuclei in the next second, but it determines for every nucleus if it survives or decays the next step, based on generated random numbers.

The exponential decay law – as expectation value – will be shown (focus only on the number of „red" atoms, and the curve representing it). The statistical deviations from the pure exponential can be well observed after a sufficiently long time, when the number of the remaining particles gets small.

Half-life

The half-life is the time during which the half of the nuclei in a radioactive substance will decay. Since the radioactive decay is a statistical process, this can be understood also only as an expectation value. The half of the number of atoms in a radioactive sample decay approximately during a half-life, but there can be some statistical fluctuations in particular cases. However, for very large number of atoms these statistical fluctuations will be negligible as compared to the number of atoms, and the actual half-life gets really close to the expectation value.

The *T* half-life can be deduced from the (2) formula, since according to the definition $N(T) = \frac{N_0}{2}$.

Substituting this in the equation we get: $\frac{N_0}{2} = N_0 \cdot e^{-\lambda \cdot T}$, from where it follows that: $\boxed{T = \frac{\ln 2}{\lambda}} \approx \frac{0{,}7}{\lambda}$.

Advice: This can be easily „checked" in the simulation: set λ = 0.01 for the decay constant, and observe the time when the number of the initial (**red**) radioactive atoms is halved (i.e. will be about 1241). You will get a value around 70 s.

Activity

The activity of a radioactive sample is the number of decays per unit time. The unit of the activity is the becquerel. It is named after Henri Becquerel French physicist, who discovered the radioactivity in 1896.

1 Bq = 1 decay/s.

The Bq is a very small unit! In the practice one uses its multiples: 1 kBq = 1000 Bq, 1 MBq = 10^6 Bq, 1 GBq = 10^9 Bq. etc.

The activity can be expressed using the exponential decay law, since it follows from the definition that the activity is $a(t) = -\frac{dN}{dt}$. Knowing the time-dependence of $N(t)$, we get by simple derivation

$$\boxed{a(t) = \lambda \cdot N(t) = \frac{\ln 2}{T} N(t)}. \qquad (3)$$

This is important, because it creates a simple relationship between the activity and the half-life (or the decay constant) and the number of radioactive nuclei. Knowing any two of them, the third can be easily determined.

Decay chain

Often happens in Nature that a radioactive element decays to another element that is itself radioactive. Thus, the „daughter" decays further and its daughter may decay too, and so on, until we reach an element that is stable at the end of the chain. Such a series of radioactive elements is called **radioactive decay chain** or radioactive series.

For example, there are 19 different isotopes in the radioactive decay chain of the ^{238}U, until the series ends at the ^{206}Pb stable isotope.

The simulation shows the temporal development of a decay chain – consisting of 5 members –, where the 5th member is a stable isotope (its decay constant is zero), and the decay constants of the first four elements can be set. The screen refreshes in every second. In the initial state there are 2482 atoms from the first element. The 5 elements of the decay chain are represented by different colours: **red** =>**blue**=>**green**=>**brown**=>**purple**.

Theoretical description

For the sake of simplicity we treat now only a radioactive decay chain consisting of 3 elements: the first two are radioactive, and the third is stable. We denote the number of atoms at a certain time *t* by $N_1(t), N_2(t), N_3(t)$ respectively, and by λ_1, λ_2 the decay constants of the first two elements. The following system of differential equations describes the temporal behaviour of the number of atoms for the different isotopes:

$$\begin{aligned} \frac{dN_1}{dt} &= -\lambda_1 \cdot N_1(t) \\ \frac{dN_2}{dt} &= -\lambda_2 \cdot N_2(t) + \lambda_1 \cdot N_1(t) \\ \frac{dN_3}{dt} &= +\lambda_2 \cdot N_2(t) \end{aligned} \qquad (4)$$

The first equation is already familiar: it describes the decay of the first element with λ_1 decay constant. Its solution can be written down (we saw it earlier): $N_1(t) = N_0 \cdot e^{-\lambda_1 \cdot t}$. Here N_0 is the initial number of atoms of the „1" material.

The second equation is a bit more complicated. The first term on the right hand side is also clear: it describes that the "2" material is also radioactive and it decays with a decay constant of λ_2. The second term on the right hand side describes the fact that the "2" material is not only decaying but it is also **created** from the „1" material. During unit time as many "2" atoms are created, as are decaying from the "1" material!

Having understood this, the meaning of the third equation should be obvious. The „3" material is not decaying (therefore the negative term is missing from the right hand side), it is only created from the „2" material.

Before solving this system of equations, we have to fix the initial conditions. We choose the following initial conditions: $N_1(0) = N_0,\ \ N_2(0) = 0,\ \ N_3(0) = 0$.

Since we know $N_1(t)$ from the solution of the first equation, we can substitute it into the second equation and we get:

$$\frac{dN_2}{dt} = -\lambda_2 \cdot N_2(t) + \lambda_1 \cdot N_0 \cdot e^{-\lambda_1 \cdot t}.$$

Since N_0 is a constant, therefore in this equation only the $N_2(t)$ function is unknown, therefore this is an inhomogeneous linear differential equation of first order. Its solution can be achieved using well-known mathematical methods. The solution satisfying the $N_2(0) = 0$ initial condition is the following:

$$N_2(t) = N_0 \cdot e^{-\lambda_1 \cdot t} \cdot \frac{\lambda_1}{\lambda_2 - \lambda_1}\left(1 - e^{-(\lambda_2 - \lambda_1)\cdot t}\right).$$

This can be rewritten as:

$$\boxed{N_2(t) = N_1(t) \cdot \frac{\lambda_1}{\lambda_2 - \lambda_1}\left(1 - e^{-(\lambda_2 - \lambda_1)\cdot t}\right)} \qquad (5)$$

***Note**: This solution is valid only if $\lambda_2 \neq \lambda_1$. If $\lambda_2 = \lambda_1 = \lambda$, then the solution is: $N_2(t) = \lambda \cdot t \cdot N_1(t)$.*

Transient radioactive equilibrium

A special case of the (5) equation is the $\lambda_2 > \lambda_1$, when the daughter nucleus decays faster than its parent. Then the exponential is negative, therefore after a sufficiently long period of time the exponential expression becomes much less than one and it can be neglected. We get:

$$\frac{N_2(t)}{N_1(t)} = \frac{\lambda_1}{\lambda_2 - \lambda_1} = \text{constant.} \qquad (6)$$

With other words, the ratio of the number of atoms (concentrations) of the members in the radioactive series becomes a constant, and does not depend on the time anymore. This is called **transient radioactive equilibrium**.

Advice: The transient radioactive equilibrium can be well observed in the simulation, if we choose for the decay constants $\lambda_2 = 2\lambda_1$ (for example $\lambda_1 = 0.07$ and $\lambda_2 = 0.14$). Based on the above equation it is to be expected that $N_2(t) = N_1(t)$ after the transient equilibrium settles. After a certain time the **red** and the **blue** curves will really go close together – except the statistical fluctuations.

The transient equilibrium can also be expressed using the **activities.** Remember, that $a(t) = \lambda \cdot N(t)$. The ratio above can be rewritten:

$$\frac{\lambda_2 \cdot N_2(t)}{\lambda_1 \cdot N_1(t)} \equiv \frac{a_2(t)}{a_1(t)} = \frac{\lambda_2}{\lambda_2 - \lambda_1} = \text{constant.}$$

(Please note that the nominator in the ratio on the right hand side differs from the one in the (6) equation!!)

Secular radioactive equilibrium

An even more specific case of the (5) equation is if $\lambda_2 >> \lambda_1$. In this case the condition for the transient equilibrium is obviously fulfilled, therefore the (6) equation is valid. However, it can be simplified even more, since now the λ_1 can be neglected in the denominator as compared to λ_2, so we get: $\frac{N_2(t)}{N_1(t)} = \frac{\lambda_1}{\lambda_2}$. This can be rewritten as: $\lambda_1 \cdot N_1(t) = \lambda_2 \cdot N_2(t)$. Remembering the notion of the activity this can also be expressed as: $a_1(t) = a_2(t)$.
This is called **secular equilibrium**.

It is easy to see, that this is true also for a radioactive decay chain with many members, if the decay constant of the very first member is much smaller than any of the following members of the chain.

In secular equilibrium the activities of the members in a radioactive decay chain are all equal!

$$\boxed{a_1(t) = a_2(t) = \ldots = a_k(t)} \qquad (7)$$

Advice: Also the secular equilibrium can be observed using this simulation. Let us set the decay constant of the first member much lower than all the others (for example 0.002 for the first, and 0.1 for all the others). After sufficiently long time it can be seen that the curves of all members (except the first) get close together (within the range of the statistical fluctuations), and they are approximately 50 times lower than that of the first member of the series.